TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 261, Number 2, October 1980

SECOND ORDER EQUATIONS OF FIXED TYPE IN REGIONS WITH CORNERS. I[1]

BY

LEONARD SARASON

ABSTRACT. A class of well-posed boundary value problems for second order equations in regions with corners and edges is studied. The boundary condition may involve oblique derivatives, and edge values may enter the graph of the associated Hilbert space operator. Uniqueness of weak solutions and existence of strong solutions is shown.

Introduction. In this paper, we study boundary value problems for second order linear partial differential equations of standard type (elliptic, parabolic, or hyperbolic), in regions with corners and edges. Our results also cover first order two-by-two systems. Only interior corners with angles normalized as $\pi/2$ have been considered, but much of the theory would appear to extend easily and with minimal changes to problems with exterior corners. This work originated as an attempt to understand formulas developed in [5] in the two-dimensional constant coefficient elliptic case.

In Chapter I, we develop the existence and uniqueness theory for elliptic problems with variable coefficients. The principal parts of the differential operator and of the boundary conditions are real. The boundary conditions for the second order equation are coercive and dissipative, in the sense that they lead to quadratic a priori estimates. We begin with the basic (Gårding) inequality (Theorem 1.1), derived by integrations by parts combined with an estimate of the boundary values of the solution u in terms of the H^1 norm of u in the interior Ω of the region under consideration. We are indebted to Stanley Osher for pointing out that Gårding's inequality is available even with corners. If the boundary conditions involve oblique derivatives, then an additional integration by parts of the boundary term is required, leading in general to a quadratic term in the edge values of u. The signature of this term determines (locally) whether or not u is to be prescribed along the edge (this effect does not occur in dissipative problems for first order systems). In §2, we use duality to show the existence of weak solutions (Theorem 2.1). In §3, we consider problems admitting the maximum principle. The existence of solutions satisfying the maximum principle when the problem is not Dirichlet, Neumann or Robin, is shown under special assumptions on the boundary condition (Theorem 3.1). We suppose that the boundary conditions involve oblique

Received by the editors May 15, 1979.
AMS (MOS) subject classifications (1970). Primary 35J25.
[1]This work was partially supported by National Science Foundation Grant MCS75-0937A01.

derivatives. Their tangential component (subtracting off a multiple of the conormal derivative) determines a vector field. If the domain and vector field are well approximable in a certain sense by domains with smooth boundaries and smooth vector fields (this excludes problems for which the edge values of u are assigned), then the approximating problems and, in the limit, the original problem, have bounded solutions satisfying the maximum principle when the data are bounded. In §5, we study the uniqueness of weak solutions, working partly with an associated first order system (Theorems 5.1 and 5.2). In the presence of multidimensional corners, we depend on the existence of bounded solutions, and this limits the applicability of our results. The methods of this section do not cover the Neumann or Robin problem, or mixtures of these. In the case of an edge, this gap is largely filled by Corollary 10.3 of **[5]** which asserts the uniqueness of solutions of such problems in the constant coefficient case (because the problem is elliptic, the result extends to a large class of variable coefficient bases). §6 extends the above results to certain equations with complex zero order terms in preparation for studying evolution equations.

In Chapter II, we extend the results of Chapter I to the parabolic and hyperbolic cases. In §7, we study parabolic problems via the standard resolvent inequality (Proposition 7.1). In §§8 and 9, we consider mixed problems for hyperbolic equations. In §8, we derive a quadratic a priori inequality (Theorem 8.1) using integrations by parts together with Gronwall's inequality. Because integration by parts is applied to the form $(\partial_t u, Lu)_\Omega$, the boundary condition will involve the t-derivative of u (here the t direction is timelike). In §9, we show the existence of strong solutions for the hyperbolic problems considered (Theorem 9.1).

I. THE ELLIPTIC CASE-EXISTENCE AND UNIQUENESS

1. The set-up: a priori inequality. We shall consider corner problems in a domain $\Omega = \Omega^{N,M} = R_+^N \times R^M$, $N \geqslant 2$, parametrized as

$$\Omega^{N,M} = \left\{ x = (y, z) : y \in R_+^N, z \in R^M \right\}.$$

The differential operator L is a second order scalar operator of the form

$$L = \sum_{j=1}^{N+M} \alpha_j \alpha_j^+ + \alpha_0, \tag{1.1}$$

where the α_j's are first order operators:

$$\alpha_j = \sum_{k=1}^{N+M} \alpha_{jk}(x)\partial_k + \alpha_{j0}(x), \qquad j = 0, \ldots, N + M, \alpha_{j0} = 0 \text{ if } j \neq 0.$$

Here α_{jk} $(k > 0)$ is in $\mathcal{C}^2$ and α_{00} is in $\mathcal{C}^1$, uniformly in $\bar{\Omega}$ and L is assumed to be uniformly elliptic:

$$\sum_j \left(\sum_k \alpha_{jk}(x)\zeta_k \right)^2 \geqslant k_0(\zeta)^2, \qquad \zeta \in R^{N+M}, x \in \bar{\Omega}, \tag{1.2}$$

with $k_0 > 0$. We assume that the principal parts of the α_j's, $j = 1, \ldots, N + M$, are orthogonal in the sense that

$$\sum_{l \neq 1}^{M} \alpha_{jl}\alpha_{kl} = 0, \qquad j \neq k,\ 1 \leqslant j, k \leqslant N + M, \tag{1.3$_a$}$$

and that α_0 has positive symmetric part:

$$\alpha_0 + \alpha_0^* = 2\alpha_{00} - \sum_{k=1}^{N+M} (\partial_j \alpha_{0j}) \geqslant \kappa_0 > 0. \tag{1.3$_b$}$$

All the coefficients above are assumed to be real valued. Denote the boundary $\partial\Omega$ by Γ, and denote the boundary faces $\partial\Omega \cap \{x\colon x_j = 0\}, j = 1, \ldots, N$, by Γ_j.

With

$$Lu - f, \tag{1.4}$$

integration by parts of the form $(u, Lu)_\Omega$ leads to the identity

$$\begin{aligned}(u, f)_\Omega = \sum |\alpha_j u|_\Omega^2 + \frac{1}{2}(u, (\alpha_0 + \alpha_0^*)u)_\Omega \\ + \sum_{0<j,k} (u, \alpha_{jk}\alpha_j u)_{\Gamma_k} - \frac{1}{2}\sum_k (u, \alpha_{0k}u)_{\Gamma_k}\end{aligned} \tag{1.5}$$

satisfied by smooth function u with bounded support. On Γ_k, set

$$D_k = \sum_j \alpha_{jk}\alpha_j - \frac{1}{2}\alpha_{0k}.$$

We consider three types of boundary condition on each Γ_k:

(a) a homogeneous Dirichlet condition $u = 0$,

(b) an inhomogeneous Robin or Neumann condition
$D_k u = \psi_k(x)u - g_k(x)$, (1.6)

(c) a more general inhomogeneous condition
$$D_k u = \sum_{l \neq k} c_{kl}\partial_l u_k + \Psi_k u - g_k.$$

Here Ψ_k and c_{kl} are uniformly in $\mathcal{C}^1$.

If case (c) occurs, we may have to assign data also along the interfaces $\Gamma_{jk} = F_j \cap \Gamma_k$, as follows. Along Γ_{jk} define

$$\varepsilon_{jk} = \begin{cases} 1 & \text{if } c_{jk} + c_{kj} < 0, \\ 0 & \text{if } c_{jk} + c_{kj} \geqslant 0, \end{cases}$$

$$\mu_{jk} = \begin{cases} 1 & \text{if } c_{jk} + c_{kj} > 0, \\ 0 & \text{if } c_{jk} + c_{kj} \leqslant 0. \end{cases}$$

We assume for simplicity that along each Γ_{jk}, $c_{jk} + c_{kj}$ is either identically zero or is bounded away from zero, and supplement (1.6) with the 'edge' condition

(d) $\varepsilon_{jk}(c_{jk} + c_{kj})u_{jk} = h_{jk}$ on Γ_{jk} where $u_{jk} = u|_{\Gamma_{jk}}$. (1.6)

Here $h_{jk} = 0$ if a homogeneous Dirichlet condition is prescribed either on Γ_j or on Γ_k. Define

$$\Gamma = \bigcup \Gamma_k, \quad \Gamma' = \left\{ \bigcup \Gamma_{jk}: \varepsilon_{jk} > 0 \right\}, \quad \Gamma'' = \left\{ \bigcup \Gamma_{jk}: \mu_{jk} > 0 \right\}.$$

Let us consider the effect on the boundary terms along Γ_k of substituting (1.6) in (1.5). In case (a) they are eliminated; in case (b), they are replaced by

$$(u, \psi_k(x)u)_{\Gamma_k} - (u, g_k)_{\Gamma_k}, \tag{1.7}$$

and in case (c), they are replaced by the sum of (1.7) and

$$\sum_{l \neq k} (u, c_{kl}\partial_l u)_{\Gamma_k}. \tag{1.8}$$

Integration by parts shows that (1.8) equals

$$-\frac{1}{2} \sum_{l \neq k} (c_{kl}u, u)_{\Gamma_{kl}} - \frac{1}{2}(u, (\partial_l c_{kl})u)_{\Gamma_k}. \tag{1.9}$$

Thus with $u_{kl} = u|_{\Gamma_{kl}}$, $u_k = u|_{\Gamma_k}$, and with the interior terms on the right-hand side of (1.5) denoted by $\langle u, u \rangle_1$, we get

$$(u, f) + \sum (u_k g_k)_{\Gamma_k} = \langle u, u \rangle_1 + \sum_k \left(u_k, \left(\psi_k - \frac{1}{2} \sum_{l \neq k} (\partial_l c_{kl}) \right) u_k \right)_{\Gamma_k}$$
$$- \frac{1}{2} \sum_{k \neq l} (u_{kl}, c_{kl} u_{kl})_{\Gamma_{kl}},$$

which we rewrite as

$$(u, f) + (u, g)_\Gamma + (u, h)_{\Gamma''} = \langle u, u \rangle_1 + (u, \phi u)_\Gamma - \tfrac{1}{2}(u, (c_{kl} + c_{lk})u)_{\Gamma'}, \tag{1.10}$$

where $h = h_{kl}$ on $\Gamma_{kl} \cap \Gamma''$. Set $\phi|_{\Gamma_k} = \phi_k$.

We want the right-hand side of (1.14) to be nonnegative with at least some interior positivity. In particular, suppose

$$\langle u, u \rangle_1 \geqslant \|\nabla u\|_\Omega^2 + K\|u\|_\Omega^2, \qquad K > 0, \tag{1.11}$$

where K may be large. If $\phi_k > 0$, $k = 1, \ldots, N$, then the right-hand side of (1.10) is positive definite and (1.15) below follows. If ϕ_k is not nonnegative, we set

$$\mu_k = \sup_{x \in \Gamma_k} |(\phi_k)_-(x)|,$$

and argue as follows. Suppose, as can always be done if K is sufficiently large, say, $K > K_0$, that we can find $\mu_k > 0$, $M_k > 0$, $k = 1, \ldots, N$, $0 < \theta_j < 1$, $j = 1, 2$, such that $\theta_1 K = K_0$ and

$$\sum \mu_k M_k < K_0, \quad \sup M_k^{-1}\mu_k < \theta_2. \tag{1.12}$$

Using the estimate

$$\|u_k\|_{\Gamma_k}^2 \leqslant 2\|u\|_\Omega \, \|\partial_k u\|_\Omega \leqslant M_k\|u\|_\Omega^2 + M_k^{-1}\|\partial_k u\|_\Omega^2, \tag{1.13}$$

we deduce from (1.10),

$$(u, f) + (u, g)_\Gamma + (u, h)_{\Gamma''} \geqslant (1 - \theta_2)\|\nabla u\|_\Omega^2 + (1 - \theta_1)K\|u\|_\Omega^2$$
$$+ \frac{1}{2}(u, |c_{kl} + c_{lk}|u)_{\Gamma'}, \tag{1.14}$$

and another application of (1.13) together with Schwarz' inequality and the estimate $\|u\|_{\Gamma''} \leqslant c\|h\|$ yields the a priori estimate

$$\|\nabla u\|_{\Omega}^2 + (K - 1)\|u\|_{\Omega}^2 + \|u\|_{\Gamma}^2 + \|u\|_{\Gamma'}^2 \leqslant C_1\left(\|f\|_{-1(\Omega)}^2 + \|g\|_{\Gamma}^2 + \|h\|_{\Gamma''}^2\right), \tag{1.15}$$

where $\| \ \|_{-1}$ is the norm in $H^{-1}(\Omega)$, the space dual to $H^1(\Omega)$ with norm

$$\|u\|_1 = \left(\|u\|^2 + \|\nabla u\|^2\right)^{1/2}, \qquad \|u\|_{-1} = \sup_{\|v\|_1 = 1} (u, v).$$

Note that if $\phi_k \geqslant 0$, $k = 1, \dots, N$, then (1.12) can be satisfied, and hence an estimate of the form (1.15) holds with $K - 1$ replaced by K, for all $K > 0$. Finally, if $g = h = 0$, then we may deduce directly from (1.14) the inequality

$$\|u\|_{\Omega} \leqslant (K - K_0)\|f\|. \tag{1.16}$$

The a priori inequalities (1.15), (1.16) are valid with any mixture of boundary conditions on the various Γ_j's as described above, under the convention that a homogeneous Dirichlet condition on Γ_k corresponds to $g_k = 0$. Thus we have

THEOREM 1.1. *Under hypotheses* (1.2), (1.3), (1.11), *and* (1.12), *solutions* $u \in \mathcal{C}_0^2(\bar{\Omega})$ *of the boundary value problem* (1.4), (1.6) *satisfy* (1.15) *and* (1.16).

COROLLARY 1.1. *The conclusion of Theorem* 1.1 *is valid if we assume that* $u \in H^2(\Omega)$ *rather than* $\mathcal{C}_0^2(\bar{\Omega})$.

2. Adjoint operator and existence of weak solutions. In order to compute adjoints, we must define an operator $\mathcal{L}$ in a Hilbert space setting, corresponding to the boundary value problem (1.4), (1.6). On Γ_k, define

$$Bu = -D_k u + \sum_{l \neq k} c_{kl}\partial_l u - \Psi_k u, \qquad k = 1, \dots, N,$$

and on $\Gamma'' \cap \Gamma_{jk}$, define

$$B''u = -(c_{jk} + c_{kj})u_{jk}.$$

Denote by Γ_- the subset of Γ on which the Dirichlet boundary condition is assigned, and set $\Gamma_+ = \Gamma \setminus \Gamma_-$. Let $H_+^1(\Omega)$ be the closure in $H^1(\Omega)$ of the set of smooth functions with bounded support and which vanish in a neighborhood of Γ_-, and define Hilbert spaces

$$\mathcal{H} = H_+^1(\Omega) \times L_2(\Gamma_+) \times L_2(\Gamma'),$$
$$\mathcal{H}^* = H_+^{-1}(\Omega) \times L_2(\Gamma_+) \times L_2(\Gamma''),$$

where H_+^{-1} is the L_2-dual (anti-dual) of H_+^1.

The operator $\mathcal{L}_0$, mapping the dense subset of $\mathcal{H}$ generated by functions $u \in \mathcal{C}_0^2(\bar{\Omega}) \cap H_+^1$ into $\mathcal{H}^*$, is defined as

$$\mathcal{L}_0\colon (u, u_\Gamma, u_{\Gamma'}) \to (Lu|_\Omega, Bu, B''u). \tag{2.1}$$

With L^* the formal adjoint of L, integration by parts leads to the identity

$$\begin{aligned}(Lu, v)_\Omega &+ (Bu, v)_\Gamma + (B''u, v)_{\Gamma''} \\ &= (u, L^*v)_\Omega + (u, B^*v) + (u, B'^*v)_{\Gamma'}, \qquad v \in \mathcal{C}_0^2(\bar{\Omega}) \cap H_2^1,\end{aligned} \tag{2.2}$$

with

$$B^*v|_{\Gamma_k\cap\Gamma_+} = -D_k v - \sum_j \partial_j(c_{kj}v) + (\Psi_k - \alpha_{0k})v,$$

$$B'^*v|_{\Gamma_{kl}} = (c_{kl} + c_{lk})v. \tag{2.3}$$

Thus with $U = (u, u_\Gamma, u_{\Gamma'})$, $V = (v, v_{\Gamma'} v_{\Gamma''})$, and

$$\mathcal{L}_0^* V = (L^*v|_\Omega, B^*v|_\Gamma, B'^*v|_{\Gamma'}), \qquad v \in \mathcal{C}_0^2(\bar{\Omega}) \cap H_+^2, \tag{2.4}$$

we have

$$(\mathcal{L}_0 U, V) = (U, \mathcal{L}_0^* V). \tag{2.5}$$

Further, since $\mathcal{L}_0^*$ has the same properties as $\mathcal{L}$, we find, if v admits $\mathcal{L}_0$, that

$$\|\nabla v\|^2 + (K-1)\|v\|^2 + \|v\|_\Gamma^2 + \frac{1}{2}\sum \varepsilon_{kl}\langle v_{\kappa l}\rangle^2$$
$$\leq C(\|L^*V\|_{-1}^2 + \|B^*V\|^2 + \|B'^*V\|^2), \tag{2.6}$$

with some fixed $C > 0$, provided $K > 0$ is sufficiently large.

From (1.15), (2.5), (2.6), and a standard application of the Riesz representation theorem shows that if $(f, g, h) \in \mathcal{K}^*$, then there exists a weak solution $U = (u, \bar{u}, u^\#) \in \mathcal{K}$, with $u \in H_+^1(\Omega)$, of the problem

$$\mathcal{L}U = (f, g, h); \tag{2.7}$$

in particular, if $v \in \mathcal{C}_0^2(\Omega)$, then

$$(f, v)_\Omega + (g, v)_\Gamma + (h, v)_{\Gamma''} = (u, L^*V)_\Omega + (\bar{u}, B^*V)_\Gamma + (u^\#, B'^*V)_{\Gamma'}. \tag{2.8}$$

We shall say that if $U = (u, \bar{u}, u^\#) \in \mathcal{K}$ satisfies (2.8), then U admits the weak extension $\mathcal{L}_\omega$ of $\mathcal{L}$, and set $\mathcal{L}_\omega U = (f, g, h)$. Thus we have proved

THEOREM 2.1. *Under hypotheses* (1.2), (1.3), (1.11) *and* (1.12), *if K is sufficiently large, that solutions $v \in \mathcal{C}_0^2(\bar{\Omega}) \cap H_+^1$ of the adjoint problem $\mathcal{L}_0^*(v, v_\Gamma, v_{\Gamma''}) = F^*$ satisfy* (2.6), *then the boundary value problem*

$$\mathcal{L}_\omega U = F \in \mathcal{K}^*$$

has a solution $U = (u, \bar{u}, u^*) \in \mathcal{K}$, i.e., U satisfies (2.8).

REMARK 2.1. We can widen the data space slightly, as follows. For functions E defined on Γ^k, set

$$\|E\|_s^2 = \|E_e\|_s^2/2, \qquad s = \pm 1/2,$$

where E_e is the even extension of E to R^{N+M-1}. Set

$$\|\bar{u}\|_{s(\Gamma)}^2 = \sum \|u_k\|_s^2, \qquad \|g\|_{s(\Gamma)}^2 = \sum \|g_k\|_s^2,$$

and let $H^s(\Gamma)$ be the closure of $\mathcal{C}^\infty(\bar{\Gamma})$ with respect to the norm $\|\ \|_{s(\Gamma)}$. Using

$$\|\bar{u}\|_{1/2(\Gamma)} \leq c\|u\|_{1(\Omega)},$$
$$(u, g) \leq \inf(\|u\|_{1/2(\Gamma)}\|g\|_{-1/2(\Gamma)}, \|u\|_{-1/2(\Gamma)}\|g\|_{1/2(\Gamma)}),$$

we see that for a slightly different class of problems, we can admit data $g \in H^{-1/2}(\Gamma)$.

3. Bounded solutions. Here we make additional assumptions which imply that smooth solutions satisfy a maximum principle. In particular, we require that there exist positive constants Ψ_+, Ψ_0, $\hat{\Psi}_0$, $\hat{c}_0$, $\hat{c}_1$, such that

$$\begin{aligned} &(a) \quad \psi_+ \geqslant \psi_k(x) \geqslant \Psi_0 > 0, \qquad k = 1, \ldots, N, \\ &(b) \quad c(x) \geqslant \hat{c}_0 > 0, \end{aligned} \tag{3.1}$$

where $c(x)$ is the zeroth order term in L, and we require that at each point of $\Gamma - \bar{\Gamma}''$, some linear combination of the associated boundary operators B_k has the form

$$\sum \omega_k B_k = \hat{C}(x) \cdot \nabla + \hat{\Psi}(x),$$

where

$$\begin{aligned} &(c) \quad \sum |w_k| \leqslant \hat{c}_1, \\ &(d) \quad \hat{\psi} \geqslant \hat{\psi}_0, \end{aligned} \tag{3.1}$$

and $\hat{C}(x)$ either points from x into $\bar{\Omega}$ or vanishes.

Let M_f, M_g and M_h denote the maximum values of $|f|$, $|g|$ and of $|h|$, which we assume are finite. Then solutions $u \in \mathcal{C}^2$ of the boundary value problem (1.4), (1.6) which tend to zero at infinity satisfy the maximum principle

$$|u| \leqslant \max\left(\psi_0^{-1} M_g,\ \hat{c}_0^{-2} M_f,\ \hat{\psi}_0^{-1} \hat{c}_1 M_g,\ \hat{c}_2 M_h\right), \tag{3.2}$$

where $\hat{c}_2 = \max(|c_{jk} + c_{kj}|^{-1})$, except that $\hat{c}_2 M_k$ is defined as zero if $M_k = 0$.

We can prove the existence of solutions satisfying (3.2) under additional hypotheses, as described in Theorem 3.1 and in Propositions 3.1 and 3.2 below; these hypotheses ensure the existence of smooth approximating problems.

THEOREM 3.1. *Assume* (3.1). *Suppose that all boundary conditions are of the forms* (1.6)(a), (b), *or that all have the form* (1.6)(c). *In the latter case, we impose the following further conditions*:

(1) $\Gamma'' = \varnothing$.

(2) *There exists* $c_{00} > 0$ *such that in a* c_{00}*-neighborhood of each* Γ_{jk}, $c_{jk} c_{kj} \leqslant - c_{00}$, *and such that in* Γ_k, $c_{00}^{-1} \geqslant |C_k| \geqslant c_{00}$, $k = 1, \ldots, N$, *with* $C_k \cdot \nabla = \sum_r c_{kr} \partial_r$.

(3) *There exist convex regions of* $\Omega^\varepsilon \supset \Omega$, $1 \geqslant \varepsilon > 0$, *with smooth boundaries* Γ^ε *and there exist nondegenerate smooth vector fields* C^ε *on* Γ^ε *such that*:

(a) $\Omega^\varepsilon \searrow \Omega$ *as* $\varepsilon \searrow 0$.

(b) *The set*

$$\Gamma_j^\varepsilon = \Gamma_j - \varepsilon e_j + \varepsilon N e_j^\cdot$$

is contained in Γ^ε, *where* $e_j^\cdot$ *is the vector whose kth component equals* $1 - \delta_{jk}$ *if* $k \leqslant N$, *and zero otherwise*.

(c) *The principal curvature of* Γ^ε *is bounded by* $(c_{00}\varepsilon)^{-1}$, $0 < \varepsilon \leqslant 1$.

(d) *Define* $\Gamma^0 = \Gamma$, $\Gamma_j^0 = \Gamma_j$, $S_0 = \bigcup_{0 \leqslant \varepsilon \leqslant 1} \Gamma^\varepsilon \setminus \bigcup_{k \neq j} \Gamma_{jk}$, *and* $C(x) = C^\varepsilon(x)$ *if* $x \in \Gamma^\varepsilon$; *on* S_0, *we require that* $C \in \mathcal{C}^1$.

(e) *There are parametrizations* ϕ^ε: $R^{N+M-1} \to \Gamma^\varepsilon$ *of* Γ^ε $(\varepsilon \geqslant 0)$, *functions* f^ε: $\Gamma^\varepsilon \to R$, *and* $\varepsilon_0 > 0$, *such that* $f^\varepsilon \in \mathcal{C}^1$ *if* $\varepsilon > 0$, $c_{00} < f^\varepsilon < c_{00}^{-1}$, *and such that*

(i) $\partial_s \phi^\varepsilon = (fC^\varepsilon) \circ \phi^\varepsilon$,

(ii) $c_{00} < \|J_{\phi^\varepsilon}\| < c_{00}^{-1}$, *where* $\|J_{\phi^\varepsilon}\|$ *is the ratio* $\omega^\varepsilon/\omega^* = |d\phi/d(s, z)|$ *of the volume form on* Γ^ε *to its pullback to* R^{N+M-1} *via* ϕ,

(iii) *The function* Φ: $(s, z, \varepsilon) \to \phi^\varepsilon(s, z)$, *is continuous in* $R^{N+M-1} \times [0, \varepsilon_0)$, *and is differentiable in* $(R^{N+M-1} \times [0, 1)) \setminus (\phi^{-1}(\bigcup \Gamma_{jk}) \times \{0\})$.

Under the above assumptions, if Ψ_0 *is sufficiently large, the boundary value problem* (1.4), (1.6) *has a unique bounded weak solution* $(u, u_\Gamma, u_{\Gamma'})$ *whenever the data* f, g *belong to* $L_\infty \cap L_2$.

Before proving Theorem 3.1, we shall describe two situations where the boundary conditions are of type (1.6)(c) and where the hypotheses can be verified:

PROPOSITION 3.1. *Suppose that the hypotheses of Theorem* 3.1 *are satisfied, except that condition* (3) *is replaced by*:

(3′) *There exists a parametrization* ϕ: $R^{N+M-1} \to \Gamma$ *of* Γ, *satisfying the following conditions*:

(a) ϕ *is piecewise smooth, and is smooth except along the preimage of* $\bigcup \Gamma_{jk}$.

(b) *With* (s, z) *coordinates of* R^{N+M-1} *as before, we require that* $\partial_s \phi = C_k \circ \phi$ *for* $(s, z) \in \phi^{-1}(\Gamma_k)$, $k = 1, \ldots, N$.

(c) *Define the formal matrix*

$$J_\phi = \begin{bmatrix} e_1 & e_2 & \cdots & e_{N+M} \\ \partial\phi_1/\partial s & \partial\phi_2/\partial s & \cdots & \partial\phi_{N+M}/\partial s \\ \partial\phi_1/\partial z_1 & \partial\phi_2/\partial z_1 & \cdots & \partial\phi_{N+M}/\partial z_1 \\ \vdots & \vdots & & \\ \partial\phi_1/\partial z_{N+M-2} & & & \partial\phi_{N+M}/\partial z_{N+M-2} \end{bmatrix}$$

and set

$$\|J_\phi\| \equiv |\det J_\phi| \equiv |\partial\phi/\partial s \times \partial\phi/\partial z \times \cdots \times \partial\phi/\partial z_{N+M-2}| \equiv |d\phi/d(s, z)|.$$

We require that in a c_{00} *neighborhood of any point in* R^{N+M-1}, *every convex combination* $\mathcal{C}(J_\phi)$ *of values of* J_ϕ *satisfies*

$$(2c_{00})^{-1} > \|\det \mathcal{C}(J_\phi)\| > 2c_{00}.$$

Then condition (3) *is also satisfied.*

PROPOSITION 3.2. *Suppose that the hypotheses of Theorem* 3.1 *are satisfied, except that condition* (3) *is replaced by*:

(3″) *We shall say that* $C_j \prec C_k$ *if* $c_{jk} < 0$ *and* $c_{kj} > 0$ *along* Γ_{jk}. *Suppose that the* C_j*'s are ordered by* $\prec$, *in that* $C_j \prec C_k \prec C_l$ *implies that* $C_j \prec C_l$. *Suppose that on each* Γ_j, *at least one coefficient* c_{jk} *never vanishes* $(1 \leqslant j \leqslant N, 1 \leqslant k \leqslant N + M)$, *and that all of the coefficients of the* C_j*'s are constant outside of some sphere, say,* $|x| > c_{00}^{-1}$. *Finally suppose that the* C_j*'s are in, say,* $\mathcal{C}^3$.

Then condition (3) *of Theorem* 3.1 *is satisfied.*

Before proving Propositions 3.1 and 3.2, we give a simple example of a vector field satisfying (3″). Let v be any vector in R^{N+M}, all of whose first N components are positive, and let Z be the hyperplane $\{x\colon\ v \cdot x = 0\}$. Define ρ to be the orthogonal projection on Z, and define ρ_j to be the restriction of ρ to Γ_j. Let w be any vector in Z which does not lie in the tangent plane of any $\rho\Gamma_{jk}$. Then the vectors

$$C_j^0 = \left|\rho_j^{-1}w\right|^{-1}\rho_j^{-1}w \quad \text{on } \Gamma_j$$

are ordered by $\prec$. We set $C_j = \varepsilon_j C_j^0$, where the ε_j's are positive numbers such that $\varepsilon_k \leqslant \varepsilon_j$ whenever $C_j \prec C_k$.

Proof of Proposition 3.1. Let $\mu\colon R^{N+M-1} \to R_+$ be a C_0^∞ function with total weight 1 and support in the unit ball:

$$\int \mu(y)\,dy = 1, \quad \mu(y) \geqslant 0, \quad \mu(y) = 0 \quad \text{if } |y| > 1,$$

and let

$$\mu_\varepsilon(y) = \varepsilon^{1-M-N}\mu(y/\varepsilon), \qquad 0 < \varepsilon.$$

Then the convolution operator $M_\varepsilon = \mu_\varepsilon^*$ is a standard mollifier.

Now we can define ϕ^ε by

$$\phi^\varepsilon y = (M_\varepsilon\phi)y - N^{1/2}\varepsilon e^{(N)},$$

where $e^{(N)}$ is the vector whose first N components are 1 and whose remaining components are zero. Then $\Gamma^\varepsilon = \phi^\varepsilon R^{N+M-1}$, and we can define the vector field C^ε on Γ^ε by setting

$$C^\varepsilon(x) = M_\varepsilon(C \circ \phi)|_{y=(\phi^\varepsilon)^{-1}x}, \tag{3.3}$$

where C is defined to equal C_j on Γ_j, $j = 1, \ldots, N$. Because $C = \partial_s\phi$ (condition (3′)(b)), and because $\partial/\partial s$ commutes with M_ε, we have $C^\varepsilon \circ \phi = M_\varepsilon\partial_s\phi = \partial_s M_\varepsilon\phi = \partial_s\phi^\varepsilon$.

Hence functions F on R^{N+M} satisfy

$$\partial(F \circ \phi) = ((C^\varepsilon \cdot \nabla)F) \circ \phi,$$

and C^ε is tangent to Γ^ε and satisfies (3)(e)(i). The remaining hypotheses of Theorem 3.1 follow easily from well-known properties of the mollifiers M_ε.

Proof of Proposition 3.2. Without loss, assume that the ordering given by $\prec$ is the natural one: $C_j \prec C_k$ if $j < k$. As above, define the field C on Γ by setting $C = C_j$ on Γ_j. The hypotheses of Proposition 3.2 imply that Γ is covered by integral curves of C originating in Γ_1, traversing no Γ_j more than once, and ending in Γ_n. Because some C_{jk} is bounded away from zero on each Γ_j, and because of the ordering of the C_j's, we have

Lemma 3.1. (a) *The lengths of the parts in* $|x| < c_{00}^{-1}$ *of the integral curves of the field* C*, are uniformly bounded.*

(b) *Every integral curve* $x(t)$ *of the field* C $(\partial_t x = C(x))$*, satisfies*

$$x(t) \in \Gamma_1, \qquad \partial_t x(t) = \lim_{|x|\to\infty,\, x\in\Gamma} C_1(x),$$

if t *is sufficiently large negative.*

With $e^{(N)}$ as in the proof of Proposition 3.1, let Z be the hyperplane $\{x: x \cdot e^{(N)} = 0\}$, let ρ denote orthogonal projection from Γ onto Z, and let ν: $R^{N+M-1} \to Z$ be an orthogonal linear map. Define ζ: $R^{N+M-1} \to \Gamma$ by $\zeta = \rho^{-1}\nu$, define ζ_ε, $\varepsilon > 0$, by

$$\zeta_\varepsilon(y) = (M_{\varepsilon/(N+M-1)}\zeta)(y) - \varepsilon e^{(N)}, \qquad y \in R^{N+M-1},$$

and set $\Gamma^\varepsilon = \zeta_\varepsilon R^{N+M-1}$. For $x \in \Gamma^\varepsilon$, define

$$C_0^\varepsilon(x) = M_\varepsilon(C \circ \zeta)|_{y = \zeta_\varepsilon^{-1}x}, \tag{3.4}$$

and define $C^\varepsilon(x)$ as the projection of $C_0^\varepsilon(x)$ along $e^{(N)}$ on $T_x(\Gamma^\varepsilon)$:

$$C^\varepsilon(x) = C_0^\varepsilon(x) - (e^{(N)} \cdot n(x))^{-1}(C_0^\varepsilon(x) \cdot n(x))e^{(N)},$$

where $n(x)$ is the interior unit normal to Γ^ε at x.

That the fields C^ε satisfy condition (3)(d) of Theorem 3.1 and are smooth and nondegenerate, and that their integral curves are perturbations of those of C ($C = C_k$ on Γ_k) follows easily from

LEMMA 3.2. *For all $\delta_0 > 0$, $1 \leqslant j < k \leqslant N$, let $\eta_{jk}^{\delta_0}$ denote the δ_0-neighborhood of Γ_{jk}:*

$$\eta_{jk}^{\delta_0} = \{x: |x - \Gamma_{jk}| < \delta_0\}.$$

Then there exists $\delta > 0$ such that

(a)(i) *if $x \in \eta_{jk}^{N\delta} \cap \Gamma \setminus \Gamma_j$, then $c_{jk} < -\delta$,*

(ii) *if $x \in \eta_{jk}^{N\delta} \cap \Gamma \setminus \Gamma_k$, $c_{kj} > \delta$,*

(b) *if $x \in \Gamma^\varepsilon \cap \eta_{jk}^{\delta}$, then*

$$(C_0^\varepsilon(x))_k - (C_0^\varepsilon(x))_j \equiv C_0^\varepsilon(x) \cdot e_{jk} > \delta, \qquad e_{jk} = e_k - e_j,$$

(*note that e_{jk} lies in Z*),

(c) *if $x \in \eta_{jk}^{\delta}$, then $|C^\varepsilon(x)| > \delta$,*

(d) *the integral curves of the field $C^\varepsilon(x)$ on Γ^ε are the projections along $e^{(N)}$ on Γ^ε of the integral curves of the projections along $e^{(N)}$ on Z of the field $C_0^\varepsilon(x)$,*

(e) *$C^\varepsilon \circ \phi_\varepsilon$ is in $\mathcal{C}^{1(\mathrm{loc})}$ as a function of (s, z, ε) in $R^{N+M-1} \times R_+ \setminus \zeta^{-1}\Gamma' \times \{0\}$, and is uniformly bounded,*

(f) *$C^\varepsilon \circ \phi_\varepsilon$ is uniformly in $\mathcal{C}^1$ in $\bigcup_{\varepsilon \geqslant 0} \phi_\varepsilon^{-1}S_0$, and C^ε is uniformly in $\mathcal{C}^1$ in S_0.*

PROOF. (a) is a consequence of the ordering of the C_j's and of their uniform continuity, (b) follows from (a) and from the fact that $C_0^\varepsilon(x)$ is a convex combination of values of $C(x)$ in $\bigcup_{j,k} \eta_{jk}^{N\delta} \cap \Gamma$, (c) follows from (b) and from the fact that $C_0^\varepsilon(x)$ is the orthogonal projection of $C^\varepsilon(x)$ on Z, (d) is obvious, (e) depends on a straightforward (omitted) calculation using the nature of the construction and the properties of mollifiers, and (f) depends on the properties of mollifiers and on the fact that in constructing C^ε, only the values of C on a single Γ_j and a single invertible linear map ρ_ε are involved.

There remains to parametrize Γ^ε, $0 < \varepsilon \leqslant \varepsilon_0$, with $\Gamma^0 = \Gamma$. Let $w_1 = \lim_{|x| \to \infty, x \in \Gamma^1} C_1(x)$, and let W_1 be the hyperplane

$$W_1 = \{x: x_1 = 0, x \cdot w_1 = 0\}.$$

Let $\rho_1 \colon R^{N+M-2} \to W_1$ be an orthogonal linear map. Along integral curves of the field C^ϵ, we require

$$\lim_{s \to -\infty} \left(s(x) - x \cdot w_1, z(x) - \rho_1^{-1}(x + \varepsilon e^{(N)}) \right) = 0,$$

together with the conditions

$$(\phi^\epsilon)^{-1}(x) = (s(x), z(x)), \qquad x \in \Gamma^\epsilon,$$
$$\partial_s \phi^\epsilon(y) = C^\epsilon(y), \qquad y \in R^{N+M-1};$$

this parametrizes Γ^ϵ as required. In particular, $\Gamma^\epsilon = \text{range } \phi^\epsilon$ because the maps ϕ^ϵ, $0 \leqslant \varepsilon \leqslant \varepsilon_0$, are homotopic to ϕ^0. Proposition 3.2 is proved.

4. Proof of Theorem 3.1. Consider first boundary conditions of the form (1.6)(c). It is convenient to define sets

$$S_k = \bigcup_{\lambda > 0} \left(\Gamma_k + \lambda(e^{(N)} - 2e_k) \right), \qquad k = 1, \ldots, N,$$

$$S_0 = \bigcup_k S_k, \quad \Gamma_\varepsilon^0 = \Gamma_\varepsilon \setminus S_0.$$

Extend the functions Ψ_k, $k = 1, \ldots, N$, smoothly to the complement Ω^c of Ω, in such a way that (3.1)(a) continues to hold, and such that

$$\Psi_k(x - \lambda e_k) = \Psi_k(x), \qquad \lambda > 0, x \in \Gamma_k,$$

and let $\Psi(x)$ denote the vector $(\Psi_1, \ldots, \Psi_N)$. We extend the data f and g similarly, f so as to be in $L_2(\Omega^\epsilon) \cap L_\infty(\Omega^\epsilon) \cap \mathcal{C}(\Omega^\epsilon)$, $\varepsilon > 0$, $\sup |f| \leqslant M_f$, and g to be in $L_2(\Gamma^\epsilon) \cap L_\infty(\Omega^\epsilon) \cap \mathcal{C}(\Gamma^\epsilon)$, $\varepsilon > 0$, with

$$g(x - \lambda \varepsilon_k) = g(x), \qquad \lambda > 0, x \in \Gamma_k, \sup |g| \leqslant M_g.$$

Let u_ε be the solution of the boundary value problem

$$Lu_\varepsilon = f \quad \text{in } \Omega^\epsilon, \tag{4.1a$_\varepsilon$}$$

with boundary condition

$$u_\varepsilon = 0 \quad \text{on } \Gamma^\epsilon \tag{4.1b$_\varepsilon$}$$

if the original boundary condition was (1.6)(a), or else

$$Du_\varepsilon = r(x)(C^\epsilon(x) \cdot \nabla)u_\varepsilon + (\Psi(x) \cdot n)u_\varepsilon + g, \qquad x \in \Gamma_\varepsilon, \tag{4.1c$_\varepsilon$}$$

where n is the inner normal to Γ^ϵ,

$$D = \sum_{j,k} \alpha_{jk} n_k \alpha_j + \alpha_{0k} n_k,$$

and $r(x) > 0$ is a piecewise smooth scale function to be chosen. For simplicity, we adopt the notation $\partial_s F$ for $\partial_s(F \circ \phi)$.

With boundary conditions of the type (1.6)(a) or (1.6)(b), the existence of u_ε is well known. With boundary condition (1.6)(c), we shall derive an a priori inequality, from which unique existence follows. In particular, integration by parts leads to the identity

$$(u_\varepsilon, f)_{\Omega^\epsilon} = \langle u_\varepsilon, u_\varepsilon \rangle_{1(\Omega^\epsilon)} + \left(u_\varepsilon, \left[\Psi(x) \cdot n + r(x) C^\epsilon(x) \cdot \nabla \right] u_\varepsilon \right)_{\Gamma^\epsilon}. \tag{4.2}$$

The boundary form on the right-hand side of (4.2) will be uniformly positive as $\varepsilon \searrow 0$ for Ψ_0 sufficiently large positive provided the form

$$(u_\varepsilon, (r(x)C^\varepsilon(x) \cdot \nabla)u_\varepsilon)_{\Gamma^\varepsilon} \tag{4.3}$$

is bounded below uniformly in ε for small positive ε.

Let us rewrite (4.3) as

$$\int u_\varepsilon^* r_\varepsilon^* \hat{J}_\varepsilon(\partial_s u_\varepsilon)\, ds\, dz, \tag{4.4}$$

where

$$\hat{J}_\varepsilon = J_\varepsilon |C^\varepsilon|\, |\partial_s \phi^\varepsilon|^{-1}$$

and where we use the general notation F_ε^* to denote $F \circ \phi_\varepsilon$ or $F^\varepsilon \circ \phi_\varepsilon$, and $J_\varepsilon = |d\phi_\varepsilon|/|ds\, dz|$. Now (4.3) is bounded below by

$$\inf_{x \in \Gamma^\varepsilon} \left[-\tfrac{1}{2} \partial_s (r_\varepsilon^* \hat{J}_\varepsilon) J_\varepsilon^{-1} \right] \|u\|_{\Gamma^\varepsilon}^2; \tag{4.5}$$

where there is no danger of ambiguity, we shall write J for J_ε. To ensure that u_ε approximately satisfies the proper boundary condition, we require $r = 1$ on S_0.

Because of hypothesis (2), one vector of the pair C_j, C_k points into Γ_{jk} along Γ_{jk}, and one points out: say, C_j points into Γ_{jk} from Γ_j, and C_k points into Γ_k from Γ_{jk}, i.e. $c_{jk} < 0$ and $c_{kj} > 0$. Since $\Gamma'' = \phi$, $c_{jk} + c_{kj} < 0$ uniformly along Γ_{jk}. Along Γ_{jk}, we have

$$\hat{J}_k / \hat{J}_j = |c_{kj}/c_{jk}|, \tag{4.6}$$

where $\hat{J}_j$ is the restriction to Γ_j of $J|C_j|/|\partial_s \phi|$. Hence J_ε decreases as we follow an integral curve $\mathcal{G}$ of the field C_ε in $\Gamma^\varepsilon \setminus \bigcup \Gamma_j^\varepsilon$. Therefore we can choose r^ε to be piecewise smooth, identically 1 in $\bigcup \Gamma_j^\varepsilon$, $r \geqslant r_0 > 0$, say, and such that $r^\varepsilon(\hat{J}_\varepsilon \circ \phi^{-1})$ decreases monotically along $\mathcal{G}$.

Since the term in brackets in (4.5) is nonnegative along $\Gamma^\varepsilon \setminus \bigcup \Gamma_j^\varepsilon$ and is uniformly bounded elsewhere, (4.3) is bounded, as desired.

Since the u_ε's are uniformly bounded in $H^1 \cap L_\infty$, we can choose a sequence $\varepsilon_j \searrow 0$ such that the restrictions u^j of u_{ε_j} to Ω converge weakly in H^1 to a limit $u \in H^1 \cap L_\infty$, and such that the boundary values of u_{ε_j} on Γ^{ε_j} converge weakly to $u|_\Gamma$ in the sense that

$$u_{\varepsilon_j}|_{\Gamma_{\varepsilon_j}^+} \rightharpoonup u_\Gamma, \quad u_{\varepsilon_j}|_{\Gamma_\varepsilon^+ \cap (\Gamma_k - \varepsilon e_k)} \rightharpoonup u_{\Gamma_k}.$$

Clearly, u is bounded and satisfies (3.2).

We want to show that u satisfies (1.6). Since u^ε is a weak solution, we have, with any $v \in C_0^\infty(R^{N+M})$,

$$(f, v)_{\Omega^{\varepsilon_j}} - \left(u_{\varepsilon_j}, L^* v\right)_{\Omega^{\varepsilon_j}} - (g, v)_{\Gamma^{\varepsilon_j}} = -\left(u_{\varepsilon_j}, B^{(*)} v\right)_{\Gamma^{\varepsilon_j}}, \tag{4.7}$$

where $B^* - D$ is the $L_2(\Gamma^{\varepsilon_j})$ adjoint of $B - D$. As $j \to \infty$, the terms on the left-hand side of (4.7) tend to

$$(f, v)_\Omega - (u, L^* v)_\Omega - (g, v)_\Gamma.$$

The term on the right-hand side of (4.7) tends to $-(u, B^*v)_\Gamma + \zeta$, where

$$\zeta = \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon^0} u_{\varepsilon_j} B^{(*)} v \, dS. \tag{4.8}$$

The measure $m(\Gamma_\varepsilon^0 \cap \operatorname{supp} v)$ is $\mathcal{O}(\varepsilon)$, $\varepsilon \searrow 0$. Since u_{ε_j} is bounded, we find that

$$\zeta = \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon^0} u_{\varepsilon_j} \partial_s(rJ|C^\varepsilon|)J^{-1} v \, dS. \tag{4.9}$$

For $z \in R^{N+M-1}$, define

$$\Gamma_z^\varepsilon = \bigcup_{s\in R} \phi_\varepsilon(s, z),$$

and for $x \in \Gamma_z^\varepsilon$, set $v_{\Gamma'}(x) = v(x')$ where $x'(x)$ is the closest point in Γ' to $\Gamma_\varepsilon^0 \cap \Gamma_z^\varepsilon$. Since $v_{\Gamma'}(x) - v(x) = \mathcal{O}(\varepsilon)$ for $x \in \Gamma_\varepsilon^0$, and since u_{ε_j} is bounded uniformly in j, we see from (4.7) that ζ is a linear functional on $v_{\Gamma'}$ in $L_1(\Gamma')$. Hence ζ can be represented as

$$\int_{\Gamma'} u_{\Gamma'} B'^* v \, dS, \tag{4.10}$$

with $u_{\Gamma'} \in L_\infty(\Gamma')$.

To show that $u_{\Gamma'} \in L_2(\Gamma')$, we rewrite (4.9) as

$$\zeta = \lim_{\varepsilon\to 0} \int dz \left(v(z) \int_{\Gamma_\varepsilon^0 \cap \Gamma_z^\varepsilon} u_{\varepsilon_j} \partial_s(rJ|C^\varepsilon|)J^{-1} \, ds \right), \tag{4.11}$$

where $v(z) = v_{\Gamma'}(x)$, x any point on $\Gamma_\varepsilon^0 \cap \Gamma_z^\varepsilon$. By Schwarz' inequality,

$$|\zeta|^2 \leqslant \lim_{\varepsilon\to 0} \left\{ \int dz(|v^2(z)|) \int_{\Gamma_\varepsilon^0 \cap \Gamma^\varepsilon} \partial_s(rJ|C^\varepsilon|)J^{-1} \, ds \right\}$$
$$\times \int \left[\int |u_{\varepsilon_j}|^2 \partial_s(rJ|C^\varepsilon|) \, ds \right] dz. \tag{4.12}$$

In (4.12), the integral $\int \partial_s(rJ|C^\varepsilon|)J^{-1} \, ds$ is bounded. Further, because of the negativity of the term

$$-\partial_s(rJ|C^\varepsilon|)J^{-1}$$

occurring in (4.5), we deduce from (4.2) by a slight modification of our previous argument that in fact the integral in square brackets on the right-hand side of (4.12) converges and is bounded independently of ε as $\varepsilon \searrow 0$. Thus ζ can be estimated by

$$\int_{\Gamma'} |v|^2 \, ds,$$

and hence is a bounded linear functional of v in $L_2(\Gamma')$. Since B'^* has a bounded inverse on Γ', it follows that $u_{\Gamma'}$, as defined by (4.10), is in $L_2(\Gamma')$.

In case the boundary conditions are homogeneous Dirichlet (1.6)(a) on some sides and Neumann or Robin (1.6)(b) on others, the only significant new problem is to define the boundary condition properly on Γ^ε. Let α be the set of integers j such that for $j \in \alpha$, the boundary condition is $u_\varepsilon = 0$, and let β be the complement of α in $\{1, \ldots, N\}$. On Γ^ε, define

$$\sin^2\theta = \sum_{j\in\alpha} n_j^2, \qquad \cos^2\theta = 1 - \sin^2\theta,$$

and let the boundary condition for u_ε be

$$\left[(\cos^2\theta)D + \sin^2\theta - \Psi(x)\cdot n\cos^2\theta\right]u_\varepsilon = g,$$

where for simplicity we first approximate g by a smooth function which vanishes on the curved part of Γ^ε. The approximating problems have smooth weak solutions which must therefore satisfy the maximum principle uniformly. Convergence as $\varepsilon \to 0$ follows essentially as above, and we omit details.

Having proved the existence of a bounded weak solution, we turn to the question of uniqueness. We want to show that any weak solution $U = (u, u_\Gamma, u_{\Gamma'})$ of the homogeneous boundary value problem with u, u_Γ and $u_{\Gamma'}$ bounded, is identically zero. Suppose that U is such a solution, and let $\Omega_\varepsilon^\delta$, $\Gamma_\varepsilon^\delta$, $\Gamma_\varepsilon^{\delta,0}$, $\Gamma_{\varepsilon,j,k}^\delta$, $\Gamma_\varepsilon^{\delta+}$ and $\Gamma_\varepsilon^{(\delta)}$ be the sets

$\Omega_\varepsilon^\delta = \Omega\setminus\{x\colon d(x, \cup_{j\neq k}\Gamma_{jk}) > \varepsilon, |x| < \delta\}$,
$\Gamma_\varepsilon^\delta = \partial\Omega_\varepsilon^\delta$,
$\Gamma_\varepsilon^{\delta,0} = \Gamma_\varepsilon^\delta\setminus\Gamma\setminus\{x\colon |x| = \delta\}$,
$\Gamma_{\varepsilon,j,k}^\delta = \partial\Gamma_\varepsilon^{\delta,0} \cap \{x\colon d(x, \Gamma_{jk}) = \varepsilon, d(x, \Gamma_l) > 2\varepsilon\} \cap \Gamma_j, j, k, l$ distinct,
$\Gamma_\varepsilon^{\delta+} = \partial\Gamma_\varepsilon^{\delta,0}\setminus \cup_{j\neq k}\Gamma_{\varepsilon,j,k}^\delta$,
$\Gamma_\varepsilon^{(\delta)} = \Gamma_\varepsilon^\delta \cap \{x\colon |x| = \delta\}$.

Since u is smooth in a neighborhood of $\Omega_\varepsilon^\delta$, we have the following analogue of (1.10):

$$\langle u, u\rangle_{1(\Omega_\varepsilon^\delta)} + \sum (u, \phi_k u)_{\Gamma_k\cap\Gamma_\varepsilon^\delta} - \frac{1}{2}\sum_{\substack{k\neq l\\ k,l<N}} (u_k, c_{kl}u_k)_{\Gamma_{\varepsilon,k,l}^\delta}$$
$$= \mathcal{O}\left(\int_{\Gamma_\varepsilon^{\delta,0}}|u|^2\,ds + \int |u|\,|\nabla u|\,dS + \int_{\Gamma_\varepsilon^{\delta+}}|u|^2\,ds\right), \tag{4.13}$$

with dS and ds Lebesgue measure on the appropriate sets.

With $\{\varepsilon_j\}$ a sequence such that $\varepsilon_j \searrow 0$ and

$$\lim_{j\to\infty} \varepsilon_j\int_{\Gamma_{\varepsilon_j}^\delta\setminus\Gamma_j^{(\delta)}}|\nabla u|^2\,dS = 0, \tag{4.14}$$

and with $(4.13)_j$ denoting (4.13) with ε_j for ε, we find that since

$$m(\Gamma_\varepsilon^{\delta,0}) + m(\Gamma_\varepsilon^{\delta+}) = \mathcal{O}(\varepsilon),$$

the first and third terms on the right-hand side of $(4.13)_j$ tend to zero as $j \to \infty$.

Let $V_{kl}^{(\varepsilon)}$ be the function on Γ_{kl} defined as

$$V_{kl}^{(\varepsilon)}(x) = u_k(x + \varepsilon e_l) - u_l(x + \varepsilon e_k)$$

with e_k and e_l the kth and lth standard unit vectors. Since

$$V_{kl}^{(\varepsilon)}(x) = u_k(x + \varepsilon e_l) + \int_0^{\pi/2}\partial_\theta u\,d\theta$$

where $\tan\theta = x_k/x_l$, we see from (4.14) that

$$\lim_{j\to\infty} \| V_{kl}^{(\varepsilon_j)} \|^2 = 0.$$

Hence, with error tending to zero as $j \to \infty$, we can replace the sum

$$\sum_{k\neq l} (u_k, c_{kl}u_k)_{\Gamma^\delta_{\varepsilon,k,l}}$$

in (4.13) with the sum

$$\frac{1}{2}\sum (u_k(\cdot + \varepsilon e_l)(c_{kl} + c_{lk})u_k(\cdot + \varepsilon e_l))_{\Gamma_{kl}}. \tag{4.15}$$

Now (4.15) is nonpositive, so that letting $j \to \infty$, we find

$$\|u\|^2_{1(\Omega^\delta)} + \sum (u, \Phi_k u)_{\Gamma_k \cap \Gamma^\delta} = \mathcal{O}\left(\int_{\Gamma^{(\delta)}} |u|\,|\nabla u|\,dS\right), \tag{4.16}$$

with Ω^δ, Γ^δ and $\Gamma^{(\delta)}$ the limits as $\varepsilon \to 0$ of $\Omega^\delta_\varepsilon$, $\Gamma^\delta_\varepsilon$ and $\Gamma^{(\delta)}_\varepsilon$.

With (1.13) replaced by the estimate

$$\|u\|^2_{\Gamma_k\cap\Gamma^\delta} \leqslant M_k\|u\|^2_{\Omega^\delta} + M_k^{-1}\|\partial_k u\|^2_{\Omega^\delta} + \|u\|^2_{\Gamma^{(\delta)}},$$

we find as an analogue of (1.14) the estimate

$$\|\nabla u\|^2_{\Omega^\delta} + \|u\|^2_{\Omega^\delta} \leqslant C\int_{\Gamma^{(\delta)}} |u|\,|\nabla u|\,dS. \tag{4.17}$$

Now let $\delta \to \infty$ through a sequence on which the right-hand side of (4.17) tends to zero, as is possible since $u \in H^1$. In the limit, we find $u = 0$, as claimed. Theorem 3.1 is proved.

5. Uniqueness of the weak solution. We have been able to prove uniqueness in case the coefficients have a translation invariance property along edges, or if the boundary condition is Dirichlet except possibly on one face. Our result is contained in Theorems 5.1 and 5.2 below.

THEOREM 5.1. *Assume the hypotheses of Theorem* 2.1, *that all boundary conditions are of type* (1.6)(c), *and that either* $N = 2$ *or the hypotheses of Theorem* 3.1 *hold. Assume that there exists* $\varepsilon_0 > 0$ *such that* $|c_{jk} + c_{kj}| > \varepsilon_0$ *along* Γ_{jk}, $j \neq k$. *Finally, assume that along each* Γ_{jk}, *all coefficients* α_{rj}, α_{rk} *are independent of* $x_{\widehat{jk}}$ *if* $r > 0$, *and that there exists* $\lambda \in \mathcal{C}^2(R^{M+N})$ *such that along each* Γ_{jk}, *the coefficients in* λB_j *and in* λB_k *of* ∂_j *and of* ∂_k *have gradients which are independent of* $x_{\widehat{jk}}$ *and which are orthogonal to* Γ_{jk}. *Then the weak operator* $\mathcal{L}_w$: $\mathcal{H} \to \mathcal{H}^*$ *is bijective.*

Note that the condition $|c_{jk} + c_{kj}| > \varepsilon_0$ ensures that weak solutions have (weak) edge values in L_2, and that we have the maximum principle and the existence of bounded solutions when $N > 2$.

THEOREM 5.2. *Assume the hypotheses of Theorem* 2.1, *and that there is at most one face on which the boundary condition is not of type* (1.6)(a), *i.e., homogeneous Dirichlet. Then the operator* $\mathcal{L}_w$: $\mathcal{H} \to \mathcal{H}^*$ *is bijective.*

REMARK. The case where all boundary conditions are of type (1.6)(b) (Neumann or Robin) depends on the construction given in [5], and we postpone its discussion till that point (cf. [5, Corollary 10.3 and Remark 10.2]).

PROOF OF THEOREM 5.1. That $R(\mathcal{L}_w) = \mathcal{H}^*$ is asserted in Theorem 2.1.

Let us consider uniqueness. Corresponding to the boundary value problem (1.1), (1.6) are weak and strong extensions $\mathcal{L}_w$ and $\mathcal{L}_s$, defined in the obvious way ($\mathcal{L}_s$ is the closure in the graph norm of $\mathcal{L}$). The uniqueness of weak solutions follows immediately from the identity

$$\mathcal{L}_w = \mathcal{L}_s, \tag{5.1}$$

which we now prove.

PROOF OF (5.1). The problem can be localized by using a partition of unity, say of Gårding type:

$$u = \sum u_j, \quad u_j = \phi_j u, \quad \sum \phi_j^2 = 1, \quad \phi_j \in C^\infty(\overline{\Omega}),$$

so there is no loss in assuming that u has small support; outside of supp u, the domain and operator may be modified at will, e.g. to ensure that the coefficients of L are nearly constant and in fact are constant outside of a small region. Assume this to have been done as needed.

In the cases $N = 0$, $N = 1$, it is well known that $\mathcal{L}_w$ and $\mathcal{L}_w^*$ are bijective. Since the range $R(\mathcal{L}_s)$ is closed, either it is all of $\mathcal{H}^*$, or there is an element $V \neq \{0, 0, 0\}$ in $H^{-1}(\Omega) \times L_2(\Gamma) \times L_2(\Gamma'')$ which is L_2-orthogonal to $R(\mathcal{L}_s)$, i.e. $\mathcal{L}_w^* V = 0$. But this contradicts the bijectivity of $\mathcal{L}_w^*$.

We shall prove $\mathcal{L}_w = \mathcal{L}_s$ next with $N = 2$, and then use induction on N for $N > 2$. Note that we can prove (5.1) either by showing directly that $\mathcal{L}_w$ and $\mathcal{L}_w^*$ have trivial kernels, as we shall do in some steps, or because $\mathcal{L}_w$ has a dissipative adjoint, namely $\mathcal{L}_s^*$, by showing that $\mathcal{L}_w^*$ has a dense range. Since $\mathcal{L}_w = \mathcal{L}_s$ is equivalent with $\mathcal{L}_w^* = \mathcal{L}_s^*$, it suffices to show that the range $R(\mathcal{L}_s)$ is dense, as we shall do in the other steps (in any case, the proofs for $\mathcal{L}$ and $\mathcal{L}^*$ are identical).

The case $N = 2$ is the more delicate one, and is the only case where our assumptions of translation invariance along edges enters. The proof when $N = 2$ is divided into four steps. First we treat the case when $M = 0$; then we permit $M > 0$ but demand that the hypotheses of translation invariance hold everywhere rather than just along the edge. The relaxation of this condition away from the edge is done in stages–first the boundary condition is allowed to vary, and then the equation as well.

Step 1. $M = 0$, $N = 2$. We need the following lemma:

LEMMA 5.1. *Let $N = 2$ and $M = 0$, and suppose that $U = (u, u_\Gamma, u_{\Gamma''}) \in \mathfrak{D}(\mathcal{L}_w)$. With $e_1 = (1, 0)$ and $e_2 = (0, 1)$, define for $\varepsilon > 0$,*

$$u_{[\varepsilon]} = (c_{12} + c_{21})^{-1}\left[c_{12}u_1(\varepsilon) + c_{21}u_2(\varepsilon)\right], \tag{5.2}$$

$$u_{(\varepsilon)} = u_1(\varepsilon) - u_2(\varepsilon), \tag{5.3}$$

$$\Gamma^\varepsilon = \{x\colon x_1 + x_2 = \varepsilon, x_1 > 0, x_2 > 0\}. \tag{5.4}$$

Then there is a special sequence $\varepsilon_j \searrow 0$ *such that*

$$\lim_{j\to\infty} \int_{\Gamma^{\varepsilon_j}} \left(|u|^2 + \varepsilon_j |\nabla u|^2\right) ds = 0, \tag{5.5}$$

$$\lim_{j\to\infty} u_{[\varepsilon_j]} = u_{12}, \tag{5.6}$$

$$\lim_{j\to\infty} u_{(\varepsilon_j)} = 0, \tag{5.7}$$

$$\lim_{j\to\infty} u_1(\varepsilon_j) = \lim_{j\to\infty} u_2(\varepsilon_j) = u_{12}, \tag{5.8}$$

$$\overline{\lim_{j\to\infty}}\ \varepsilon^{-1} \int_{\Gamma^{\varepsilon_j}} u^2\, ds < \infty. \tag{5.9}$$

PROOF. Note that (5.5) holds for almost all $\varepsilon > 0$; this follows directly from the fact that $u \in H^1(\Omega)$, and (5.8) follows from (5.6) and (5.7). Note also that if $t > 0$, then $u(te_2) = u_1(t)$, and $u(te_1) = u_2(t)$. Define

$$\Omega_\varepsilon = \Omega \setminus \{x : x_1 + x_2 < \varepsilon\}.$$

Since $U \in \mathfrak{D}(\mathfrak{L}_s)$ locally, away from the origin, u satisfies the analogues of (1.10) and (1.15) in Ω_ε, and U satisfies the analogue of (2.2) in Ω_ε, with v any test function. Except for terms that obviously tend to zero as $\varepsilon_j \searrow 0$ along a special sequence, the difference between (2.2) and its analogue in Ω_ε (with no special boundary conditions substituted along Γ^ε) is just

$$(c_{12} + c_{21})u_{12}v_{12} - c_{12}u_1(\varepsilon)v(0, \varepsilon)$$
$$- c_{21}u_2(\varepsilon)v(\varepsilon, 0) - \int_{\Gamma^\varepsilon} (uD_\nu v - vD_\nu u)\, ds, \tag{5.10}$$

with D_ν the conormal derivative along Γ^ε. Choosing $v \equiv 1$ in a neighborhood of the origin, we find

$$\lim_{j\to\infty} (c_{12} + c_{21})(u_{12} - u_{[\varepsilon_j]}) + \int_{\Gamma^{\varepsilon_j}} D_\nu u\, ds = 0. \tag{5.11}$$

We estimate

$$\left| \int_{\Gamma^\varepsilon} D_\nu u\, ds^2 \right| < \varepsilon\sqrt{2} \int_{\Gamma^\varepsilon} |D_\nu u|^2\, ds,$$

which tends to zero because of (5.5). Since $c_{12} + c_{21} \neq 0$, this implies (4.6). Now represent $u_{(\varepsilon)}$ as

$$u_{(\varepsilon)} = \frac{1}{\sqrt{2}} \int_{\Gamma^\varepsilon} (\partial_2 - \partial_1)u\, ds; \tag{5.12}$$

application of Schwarz' inequality and (5.5) to (5.12) yields (5.7).

To prove (5.9), let us represent u along Γ^ε as

$$u(x_1, \varepsilon - x_1) = u_1(\varepsilon) + \int_0^{x_1} (\partial_1 - \partial_2)u(t, \varepsilon - t)\, dt$$
$$\stackrel{\text{def}}{=} u_1(\varepsilon) + u^{(2)}. \tag{5.13}$$

Because of (5.8), $u_2(\varepsilon_j)$ is uniformly bounded as $j \to \infty$, and hence

$$\int_{\Gamma^{\varepsilon_j}} |u_1(\varepsilon)|^2 \, ds = \mathcal{O}(\varepsilon_j). \tag{5.14}$$

Moreover,

$$|u^{(2)}(x_1, \varepsilon - x_1)|^2 \leqslant \sqrt{2}\, \varepsilon \int_{\Gamma^\varepsilon} |\nabla u|^2 \, ds, \qquad 0 < x_1 < \varepsilon,$$

so that, by (4.5),

$$\int_{\Gamma^{\varepsilon_j}} |u^{(2)}(x)|^2 \, ds \leqslant \sqrt{2}\, \varepsilon_j^2 \int_{\Gamma^{\varepsilon_j}} |\nabla u|^2 \, ds = \mathcal{O}(\varepsilon_j). \tag{5.15}$$

Lemma 5.1 is proved.

We now claim that u satisfies (1.10), hence also (1.15), which implies (5.5). To see it, consider, in Ω_{ε_j}, the analogue of (1.10), with boundary form along Γ^{ε_j} given as

$$\int_{\Gamma^{\varepsilon_j}} u D_\nu u \, ds. \tag{5.16}$$

Because of (5.8), it suffices for our purpose to show that (5.16) tends to zero as $j \to \infty$, as follows easily from (5.5) and (5.9).

Step 2. *"Cylindrical" problems*. Here $N = 2$ and $M > 0$. We assume that all coefficients α_{r1}, α_{r2}, c_{12}, and c_{21} are independent of $x_3, \ldots, x_{M+2}$. The problem is reduced to the previous case by applying to the weak solution a mollifier in the "cylindrical" variables $x_3, \ldots, x_{M+2}$. Because of Step 1 and the properties of the smoothing operator, the partially smoothed solution approximately solves the original problem and permits formal integration by parts, thus satisfying the analogue of (1.19). In the limit as the support of the mollifier shrinks to zero, we get the uniqueness of the original weak solution, and, because the adjoint problem is of the same type, (5.1). Details are left to the reader.

Step 3. *More general boundary condition*, $N = 2$. Suppose that the hypotheses of Theorem 5.1 are satisfied with $B = B_0 + B_1$, where L, B_0, and B'' together satisfy the conditions of Step 2, and $(B_1)_{\Gamma_i} = \sum b_{ij}\partial_j$, $i = 1, 2$, with $b_{ij} = 0$ on Γ_{12} and $b_{ij} \in C^1$. Assume that the coefficients α_{r1}, α_{r2}, c_{12}, and c_{21}, which in Step 2 were assumed to be independent of x_{12}, are actually constant, along with α_{01} and α_{02}. As is justified by the fact that (5.4) is a local question, we assume that B_1 has small support and coefficients bounded in C^1, independent of the size of that support. We shall prove the existence of strong solutions directly by construction; together with the same result for the adjoint problem, this implies uniqueness for both, and hence (5.1).

We construct strong solutions of the perturbed problem by iterations:

$$\begin{aligned} Lu^{n+1} &= f, \\ B_0 u^{n+1} &= g - B_1 u^n, \\ B'' u^{n+1} &= h, \quad u^1 = 0, \end{aligned} \tag{5.17}$$

where $(u^{n+1}, u_\Gamma^{n+1}, u_{\Gamma'}^{n+1})$ is a strong solution of (5.17). That strong solutions u^{n+1} exist follows from the assumption that the problem (L, B_0, B'') falls under Step 2.

Denote the unperturbed and the perturbed problems by $\mathcal{L}_0 U \leqslant F$ and $(\mathcal{L}_0 + \mathcal{L}_1)U = F$, and let p_j denote projection on the jth component in $\mathcal{H}$, $j = 1, 2$, i.e.,

$$p_1(u, u_\Gamma, u_{\Gamma'}) = u, \qquad p_2(u, u_\Gamma, u_{\Gamma'}) = u_\Gamma.$$

Convergence of the iterations (5.17) to a strong solution of the perturbed problem follows from the existence of an a priori estimate of the form

$$\|B_1(u^{n+1} - u^n)\|_\Gamma \leqslant \theta \|B_1(u^n - u^{n-1})\|_\Gamma, \qquad \theta < 1,$$

i.e.,

$$\|B_1 p_2 \mathcal{L}_0^{-1}(0, g, 0)\|_\Gamma \leqslant \theta \|g\|_\Gamma, \qquad g \in L_2(\Gamma). \tag{5.18}$$

We shall apply a standard coercive estimate (cf. **[1]**, **[3]**) for solutions of the half-space problem, namely

$$\|w\|_{2(\Omega')} \leqslant C(\|Lw\|_{\Omega'} + \|Bu\|_{1/2(\Gamma)} + \|w\|_{\Omega'}), \tag{5.19}$$

where Ω' is a half space with boundary Γ, for functions w with support in a fixed hemisphere. Because of the dissipative nature of our problem, the term $\|w\|_\Omega$ on the right-hand side of (5.19) can be dropped.

Let Λ be the pseudo-differential operator in $x_{\hat{1}}$, given as

$$\Lambda = \left(1 - \sum_{j>1} \partial_j^2\right)^{1/2}.$$

Because of the assumed partial translation invariance of L and of B_0, both operators approximately commute with powers of Λ, and (5.19) can easily be converted to

$$\|\Lambda^{-1} w\|_{2(\Omega)} \leqslant C\big(\|\Lambda^{-1} Lw\|_\Omega + \|\Lambda^{-1} B_0 w\|_{1/2(\Gamma)} + \|w\|_\Gamma\big). \tag{5.20}$$

Let us substitute in (5.20) the functions $u_\varepsilon = \Phi^\varepsilon u$, $0 < \varepsilon \leqslant 1$, where Φ^ε is a $\mathcal{C}_0^\infty(R^{M-2})$ function satisfying $\Phi^\varepsilon(x) = \mu(x_1/\varepsilon)\rho(x_{\hat{1}})$ with μ and ρ in $\mathcal{C}_0^\infty$, $\mu, \rho \geqslant 0$, supp $\mu = [\frac{1}{2}, 1]$, diam supp $\rho = 1$, and such that both μ and ρ are positive on the interior of their supports. Now if $\mathcal{L}_0 u = (0, g, 0)$, then

$$\|\Lambda^{-1/2} L \Phi^\varepsilon u\| \leqslant C(\varepsilon^{-3/2}\|u\| + \varepsilon^{-1/2}\|\nabla u\|) \leqslant C\varepsilon^{-3/2}\|g\|, \tag{5.21}$$

$$\|\Lambda^{-1/2} B_0 \Phi^\varepsilon u\|_\Gamma \leqslant C\|g\| \tag{5.22}$$

with $C > 0$ independent of ε. Thus (4.20) implies

$$\|\Lambda^{-1/2} \Phi^\varepsilon u\|_{2(\Omega)} \leqslant C_1 \varepsilon^{-3/2}\|g\|, \tag{5.23}$$

with $C_1 > 0$ independent of ε. Because of our hypothesis, the coefficients of B_1 are $\mathcal{O}(\varepsilon^2)$ on supp Φ^ε, and hence

$$\|B_1 \Phi^\varepsilon u\|_{\Gamma_1} \leqslant C_2 \varepsilon^{-1/2}\|g'\|, \qquad \varepsilon \to 0. \tag{5.24}$$

Thus if B_1 has sufficiently small support, (5.18) obtains with $\theta < 1$.

Step 4. $N = 2$, *the general case*. Here we hold B fixed and perturb L. Let $L = L_0 + L_1$, where (L_0, B, B'') satisfy the conditions of Step 3. We construct strong solutions of the perturbed problem by the continuation method, using successive iterative steps. Thus we need only show that the iterations $u^1 = 0$,

$$\begin{aligned} L_0 u^{n+1} &= f - L_1 u^n, \\ B u^{n+1} &= g, \\ B'' u^{n+1} &= h, \end{aligned} \tag{5.25}$$

converge when the coefficients of ∂_{11} and of ∂_{22} in L_1 vanish along Γ_{12} and all the coefficients of L_1 are small with small support. In the present case, convergence of the iterations (4.25) to a strong solution of the perturbed problem will follow from an estimate of the form

$$\|L_1(u^{n+1} - u^n)\|_{-1} \leqslant \theta \|L_1(u^n - u^{n-1})\|_{-1}, \tag{5.26}$$

i.e.,

$$\|L_1 p_1 \mathfrak{L}_0^{-1}(w, 0, 0)\|_{-1} \leqslant \theta \|w\|_{-1}, \qquad w \in H^{-1}(\Omega), \tag{5.26$'$}$$

with $\theta < 1$. By adding a large constant to L (which leaves (5.1) invariant), we find from (1.19),

$$\|p_1 \mathfrak{L}_0^{-1}(w, 0, 0)\|_1 \leqslant C^0 \|w\|_{-1}$$

where C^0 is arbitrarily close to $C_1^{1/2}$, and C_1 is the constant in (1.15). We want therefore to show that if $\beta \in \mathcal{C}^1$, and if either $\beta = 0$ on Γ, or if j_1 or $j_2 > 2$, then

$$\beta \partial_{j_1} \partial_{j_2} \colon H^1(\Omega) \to H^{-1}(\Omega)$$

is a bounded operator with bound estimated by the C^1 norm of β. Since the β's we will need support arbitrarily close to Γ_{12}, we can arrange uniformly for the $\mathcal{C}^1$ norm of β to be arbitrarily close to the maximum norm of $|\nabla \beta|$.

Because $\mathfrak{L}_0$ is the strong operator, it will suffice to consider the effect of $\beta \partial_{j_1} \partial_{j_2}$ on functions $u \in C_0^\infty(R^{2+M})$. Now

$$\|\beta \partial_{j_1} \partial_{j_2} u\|_{-1} = \sup \|v\|_1^{-1} |(\beta \partial_{j_1} \partial_{j_2} u, v)_\Omega|.$$

If j_1 or $j_2 > 2$, we integrate the inner product by parts with respect to that variable, and get the desired result immediately. Suppose $j_1 \leqslant 2$ and $j_2 \leqslant 2$. Then an integration by parts gives

$$(\beta \partial_{j_1} \partial_{j_2} u, v)_\Omega = -(\partial_{j_1} u, \partial_{j_2} \beta v)_\Omega - (\partial_{j_1} u, \beta v)_{\Gamma_{j_2}}. \tag{5.27}$$

Here the first term on the right is bounded as desired. Let us consider the second term. Without loss, suppose that $j_2 = 1$, and define

$$\Omega^{(1)} = \{x \colon x_1 > 0\}, \quad \Gamma^{(1)} = \{x \colon x_1 = 0\},$$

and define E_e (E_e^-) to be the even extension operators from $L_2(\Omega)$ $(L_2(\Gamma_1))$ to $L_2(\Omega^{(1)})$ $(L_2(\Gamma^{(1)}))$, and define E_0 (E_0^-) to be the corresponding null extension operators, e.g.,

$$(E_e u)(x_1, -x_2, x_{\widehat{12}}) = u(x), \qquad x \in \Omega,$$

$$(E_0 u)(x) = 0, \qquad x_2 < 0.$$

Since E_e and βE_0 are bounded operators from $H^1(\Omega)$ to $H^1(\Omega^1)$, we have, also using the boundary condition if $j_1 = 1$,

$$\|E_e^- \partial_{j_1} u\|_{-1/2(\Gamma^{(1)})} \leq c\|u\|_{1(\Omega)},$$
$$\|E_e^- \beta v\|_{1/2(\Gamma^{(1)})} \leq c\|v\|_{1(\Omega)}. \tag{5.28}$$

Because $E_0^- \beta_v = 0$ if $x_2 < 0$ and because the shift operator is continuous in $H_\pm^{1/2}$, (4.28) implies that the second term on the right in (4.27) equals $(E_e^- \partial_{j_1} u, E_0^- \beta v)_{x_1 = 0}$ and is estimated by $c\|u\|_{1(\Omega)}\|v\|_{1(\Omega)}$, as desired. This concludes the proof of (5.1) when $N = 2$.

We turn to the case $N = N_0 > 2$ and make the inductive hypothesis that (4.4) holds with $N = 2, 3, \ldots, N_0 - 1$.

Let $U = (u, u_{\Gamma'}, u_{\Gamma''})$ be the bounded solution constructed in §3, and suppose that $V = (v, v_{\Gamma'}, v_{\Gamma''}) \in \mathfrak{D}(\mathfrak{L}_w^*)$. Let $\rho: R \to [0, 1]$ be smooth and satisfy

$$\rho(t) = \begin{cases} 1, & t \geq 1, \\ 0, & t \leq \frac{1}{2}, \end{cases}$$

and set $\rho_\varepsilon(x) = \rho(\varepsilon^{-1}|x|)$, $\varepsilon > 0$. By the inductive hypothesis, $\rho_\varepsilon U \in \mathfrak{D}(\mathfrak{L}_s)$ and $\rho_\varepsilon V \in \mathfrak{D}(\mathfrak{L}_s^*)$, since ρ_ε vanishes near

$$\Gamma_0 \overset{\text{def}}{=} \Gamma \cap \{x: x_1 = x_2 = \cdots = x_N = 0\}.$$

As observed above, there is no loss in assuming that U has bounded support, in which case the same can be assumed of V. Now

$$\mathfrak{L}_w(\rho_\varepsilon U) = \rho_\varepsilon \mathfrak{L}_w U + [\mathfrak{L}_w, \rho_\varepsilon] U, \tag{5.29}$$

and, since Γ_0 is a lower dimensional submanifold of $\bar{\Gamma}' \cup \bar{\Gamma}''$, we have

$$\rho_\varepsilon \mathfrak{L}_w U \to \mathfrak{L}_w U \quad \text{and} \quad \rho_\varepsilon U \to U, \quad \text{as } \varepsilon \searrow 0.$$

Thus, by letting $\varepsilon \searrow 0$ in (5.29), we shall have established

$$(\mathfrak{L}_w U, V) - (U, \mathfrak{L}_w^* V) = 0 \tag{5.30}$$

once we show that

$$\lim_{\varepsilon \searrow 0} \big([\mathfrak{L}_w, \rho_\varepsilon] U, V\big) = 0, \tag{5.31}$$

and (5.30) implies that U lies in the domain of the adjoint $\mathfrak{L}_s$ of $\mathfrak{L}_w^*$.

There remains to prove (5.31). Write

$$[\mathfrak{L}_w, \rho_\varepsilon] = \sum_{j>0} (S_j \rho_\varepsilon)\partial_j + (T\rho_\varepsilon),$$

where T is a second order operator and the S_j's are first order operators. Then

$$\big([\mathfrak{L}_w, \rho_\delta] U, V\big) = \left(\left[\sum_{j>0} (S_j \rho_\varepsilon)\partial_j u + (T\rho_\varepsilon) u\right], v\right)_\Omega$$
$$+ \sum_k \left(\sum_{l \neq k} c_{kl}(\partial_l \rho_\varepsilon) u, v\right)_{\Gamma_k}.$$

Integrating by parts, we can write this as

$$([\mathcal{L}_w, \rho_\varepsilon]u, v) = \left(u, \left[-\sum_{j>0}(S_j\rho_\varepsilon) + (T\rho_\varepsilon)\right]v\right) + \mathcal{O}((|\nabla\rho_\varepsilon|\,|u|, |v|)_\Gamma), \qquad (5.32)$$

and (5.31) follows from (5.32) if we apply Schwarz' inequality to each term on the right-hand side of (5.32) and use the boundedness of u together with the estimates

$$m(\operatorname{supp} \nabla\rho_\varepsilon) = \mathcal{O}(\varepsilon^3) \quad \text{in } \Omega,$$
$$m(\operatorname{supp} \nabla\rho_\varepsilon) = \mathcal{O}(\varepsilon^2) \quad \text{in } \Gamma,$$
$$|\nabla\rho_\delta| = \mathcal{O}(\varepsilon^{-1}), \qquad |\nabla\nabla\rho_\varepsilon| = \mathcal{O}(\varepsilon^{-2}),$$

and

$$\int_{\operatorname{supp}(\nabla\rho_\varepsilon)} |v|^2\, dX = \mathcal{O}(\varepsilon).$$

The last estimate follows from the facts that $v \in H^{-1}(\Omega)$, which implies

$$\int_{x_1=\delta} |v|^2\, dX/dx_1 \leqslant C\|v\|_1^2, \qquad \delta > 0,$$

and that $\operatorname{supp}(\Sigma\rho_\varepsilon) \subset \{x\colon 0 \leqslant x_1 \leqslant \varepsilon\}$.

Sketch of proof of Theorem 5.2. As before, Theorem 2.1 asserts that $R(\mathcal{L}_w) = \mathcal{H}^*$. We shall show directly that $\mathcal{L}_w$ is one-to-one; together with the same result for the adjoint problem, this implies that $T_w = T_s$. The method is as follows. Approximate the domain by domains $\Omega_\varepsilon = \Omega$ and which have piecewise planar boundaries, as follows. Let Z be the set of $N + M$-tuples α whose components α_j are 1 or 0, with at least two entries equal to 1, and with $\alpha_j = 0$ if $j > N$. Let

$$\Omega_\varepsilon = \bigcap_{\alpha \in Z} \{x \in \Omega\colon x \cdot \alpha > \varepsilon\}, \quad \Gamma_\varepsilon = \partial\Omega_\varepsilon, \quad \Gamma_\varepsilon^0 = \Gamma \setminus \Gamma_\varepsilon,$$

and suppose that $\mathcal{L}_w u = 0$. Because u vanishes on at least $N - 1$ faces of Γ, we can represent u on any planar component $\Gamma_\varepsilon^\alpha$ of Γ_ε^0 as

$$u(x) = \int \partial_s u(x)\, ds,$$

with $\partial_s u$ some directional derivative within Γ_ε^0 of u, and where u vanishes at the initial point of the integral. Thus we find

$$\|u\|_{\Gamma_\varepsilon^0} \leqslant C\varepsilon\|\partial_s u\|_{\Gamma_\varepsilon^0},$$

which implies

$$\left|\int_{\Gamma_\varepsilon^0} uD_\nu u\right| \leqslant C\varepsilon\|\nabla u\|_{\Gamma_\varepsilon^0}^2, \qquad (5.33)$$

where D_ν is the conormal derivative along Γ_ε^0. As in the proof of Theorem 5.1, we conclude that since the right-hand side of (5.33) tends to zero with ε, so does the left-hand side, and hence that in the limit, formal integration by parts is justified, i.e., u satisfies (1.19) and (1.20). Thus $u \equiv 0$, as claimed.

6. Extensions to $L + \lambda$. In dealing with evolution equations, we shall want the following 'complex' extension of most of the results of §§1–5.

THEOREM 6.1. *Let the hypotheses of any of Theorems* 1.1, 2.1 *or* 5.1 *hold. Consider the same boundary value problem, but with L replaced by* $L + \lambda$, $\operatorname{Im}\lambda \geqslant 0$. *Then, with the following changes, the conclusion of the theorem remains valid*:

(a) *In* (1.15), (1.16), *and* (2.6), *K is replaced by* $K + \operatorname{Re}\lambda$,

(b) *In* (2.8), *the inner product is a complex one. We note the extended estimates by* $(1.15)_c$, $(1.16)_c$ *and* $(2.6)_c$.

SKETCH OF PROOF. Denote the complex valued version of the above theorems by Theorems 1.1_c, 2.1_c, 5.1_c. To prove Theorem 1.1_c, we repeat the proof of Theorem 1 using the real parts of complex inner products in place of real inner products. To prove Theorem 2_c we repeat the proof of Theorem 2, but with complex inner products, and with K in (2.6) replaced by $K + \operatorname{Re}\lambda$. The proof of Theorem 5.1 remains the same, but the existence of strong solutions is derived from the result for $\lambda = \lambda_0$ real by the 'continuation' method, that is, by successive small perturbations of λ; at each step, the problem is solved by iterations of the form

$$(L - \lambda_r)^{r,s+1} = (\lambda_r - \lambda_{r+1})u^{r,s} + f, \qquad r = 0, 1, \ldots, Q < \infty, \lambda_Q = \lambda.$$

II. EVOLUTION EQUATIONS

7. The parabolic case. For simplicity, we consider in this chapter only time independent equations and boundary conditions. Let L, $\mathfrak{L}$, B, B'', Ω, Γ, Γ', Γ'' be as described in Chapter I, and assume the hypotheses of Theorems 1.1, 2.1 and 5.1. Let $\Omega^+ = R_+ \times \Omega$ have coordinates (t, x), $t \in R_+$, and let Ω^+ have boundary

$$\partial\Omega^+ = \Gamma^+ = \Gamma_0^+ \cup \Gamma_1^+,$$

where $\Gamma_1^+ = R_+ \times \Gamma$, $\Gamma_0^+ = \{0\} \times \Omega$, and let

$$\Gamma'^+ = R_+ \times \Gamma', \qquad \Gamma''^+ = R_+ \times \Gamma''.$$

Let $\mathfrak{L}_0$ be the restriction of $\mathfrak{L}$ to functions satisfying homogeneous boundary conditions, with domain considered as the first component of $D(\mathfrak{L})$, i.e., $D(\mathfrak{L}_0) \subset L_2(\Omega)$. As a consequence of Theorems $(1.1)_c$, $(2.1)_c$, $(5.1)_c$, and in particular the estimate $(1.16)_c$, we have

PROPOSITION 7.1. *If* $\operatorname{Re}\lambda < K - K_0$, *then the resolvent*

$$R_\lambda(\mathfrak{L}_0) = (\lambda - \mathfrak{L}_0)^{-1}$$

satisfies the inequality

$$\|R_\lambda(\mathfrak{L}_0)\| \leqslant (K - K_0 - \operatorname{Re}\lambda)^{-1}. \tag{7.1}$$

Consequently the operator

$$\mathfrak{L}_{0,z} = -\mathfrak{L}_0 - z$$

generates a strongly continuous semigroup of operators provided $\operatorname{Re} z < K - K_0$.

The solution of the analogous inhomogeneous mixed problems

$$
\begin{aligned}
(\partial_t + L)u &= f \quad \text{in } \Omega^+, \\
Bu &= g \quad \text{on } \Gamma^+, \qquad B''u = h \quad \text{on } \Gamma''^+, \\
u &= u_0 \quad \text{on } \Gamma_0^+,
\end{aligned}
\tag{7.2}
$$

can be shown to exist by an extension of the methods of Chapter I.

Integration by parts as in §1 leads to the a priori estimates

$$
\begin{aligned}
&\|\nabla_x u\|^2_{\Omega^+} + (K-1)\|u\|^2_{\Omega^+} + \|u\|^2_{\Gamma_1^+} \\
&\quad \leqslant C_1\Big(\int_0^\infty \|f(t,\cdot)\|^2_{-1(\Omega)}\,dt + \|g\|^2_{\Gamma_1^+} + \|h\|^2_{\Gamma''^+} + \|u_0\|^2_{\Gamma_0^+},
\end{aligned}
\tag{1.15$^+$}
$$

$$
\|u\|^2_{\Omega^+} \leqslant (K-K_0)\|f\|^2_{\Omega^+}. \tag{1.16$^+$}
$$

Weak solutions are defined in the obvious way: if $\partial_t + L = M$ and the corresponding weak operator is denoted by $\mathfrak{M}_\omega$, then the domain of $\mathfrak{M}_\omega$ is $L_2(R_+; \mathcal{H})$, and its range is $L_2(R_1, \mathcal{H}^*) \times L_2(\Gamma_0^+)$. The uniqueness of weak solutions is settled by first smoothing in the t direction, with a shift:

$$
u_\varepsilon = \varepsilon^{-1}\int j((t-t'-2\varepsilon)/\varepsilon)u(t',x)\,dt',
$$

where $j \in C_0^\infty(R)$, with range in $\overline{R}_+$, support in $[-1, 1]$, and integral equal to 1. If u vanishes for $t < 0$, so does u_ε, and thus, because of the translation invariance in t of the differential operator and of the remaining boundary operators, u_ε approximates u in the graph form. Since u_ε is smooth in the t direction, $u_\varepsilon(t,\cdot)$ admits $\mathfrak{L}_\omega$ for all t; this reduces the problem to showing the validity of the energy estimates for elements in the domain of $\mathfrak{L}_\omega$. But this follows from the results of §§3 and 5.

8. The hyperbolic case; a priori estimate. The equation has the form

$$
\mathfrak{M}u = -\tau^*\tau + \sum_{j=1}^{N+M} \alpha_j^*\alpha_j + \alpha_0 = f, \tag{8.1}
$$

where the α_j's satisfy the same conditions as in §1, except that they include t-differentiations:

$$
\alpha_j = \alpha_{j0}\partial_t + \sum_{k=1}^{N+M} \alpha_{jk}(x)\partial_k + \alpha_{j0}(x),
$$

$$
\tau = \partial_t + \sum_{k=1}^{N+M} \tau_k(x)\partial_k + \tau_0(x)
$$

satisfies the same conditions as the α_j's, $j > 0$, and the vectors τ, $\alpha_j = (\tau_1, \ldots, \tau_{N+M})$, $(\alpha_{j1}, \ldots, \alpha_{j(N+M)})$, $j = 1, \ldots, N+M$, are mutually orthogonal. Note that all coefficients are assumed to be independent of t.

We assume further that $\mathfrak{M}$ is hyperbolic with the hyperplanes $\{t\} \times R^{N+M}$ uniformly spacelike for $t \geqslant 0$, and that the time-reduced equation is uniformly elliptic in Ω^+. For convenience, the hyperbolicity of $\mathfrak{M}$ will be assumed in the following form:

Let τ^1, α_j^1 be the principal parts of τ, α_j, $\alpha = 1, \ldots, N + M$. Then, the form

$$Q(t, x; \partial_t u, \nabla_x u) = |u_t|^2 + \sum_{j=1}^{N+M} |\alpha_j^1 u|^2 - \sum_{j=1}^{N+M} \tau_j^2 |\partial_j u|^2 + 2 \sum_{j=1}^{N+M} \alpha_{j0}(\partial_t u)(\alpha_j^1 u)$$

is strictly and uniformly positive in Ω^+:

$$Q(t, x, \partial_t u, \nabla_x u) \geqslant q_0(|\partial_t u|^2 + |\nabla_x u|^2), \qquad q_0 > 0,\ q_0 \text{ fixed.}$$

Let $\Omega^T = [0, T] \times \Omega$, $\Gamma^T = [0, T] \times \Gamma$, $\Gamma_k^T = [0, T] \times \Gamma_k$, and $\Gamma_{jk}^T = [0, T] \times \Gamma_{jk}$, $T > 0$, and let $\Omega_T = \{T\} \times \Omega$, $T \in R$. We derive an energy estimate for solutions of the mixed problem for $\mathfrak{M}$ by integrating by parts the form $2(\partial_t u, \mathfrak{M}u)_{\Omega^T}$, with $u \in \mathcal{C}_0^2(R^{N+M})$:

$$\begin{aligned}
2(\partial_t u, \mathfrak{M}u)_{\Omega^T} &= 2\Big\{ (\partial_t u, \partial_{tt} u)_{\Omega^T} + \sum_{j>0} (\partial_t u, \alpha_j^* \alpha_j u + \alpha_0 u)_{\Omega^T} \\
&\quad - \Big(\partial_t u, \Big(\sum_{j>0} \tau_j \partial_j\Big)^* \sum_{j>0} \tau_j \partial_j u\Big)_{\Omega^T} - 2\Big(\partial_t u, \sum_{j>0} \tau_j \partial_j \partial_t u\Big)_{\Omega^T} \Big\} \\
&= \int_0^T \partial_t \Big\{ (u_t, u_t)_{\Omega_t} + \sum_{j>0} (\alpha_j u, \alpha_j u)_{\Omega_t} - \sum_{j>0} (\tau_j \partial_j u, \tau_j \partial_j u)_{\Omega_t} \Big\}\, dt \\
&\quad + \sum_{k=1}^{N} \Big(\partial_t u, \Big(\sum_{j>0} \partial_{jk} \partial_j - 2\tau_k \tau - \alpha_{0k}\Big) u\Big)_{\Gamma_k^T} \\
&\quad + \Big(\partial_t u, \Big(\sum_{j>0} \alpha_{j0} \alpha_j - \alpha_{00}\Big) u\Big)\Big|_{\Omega_0}^{\Omega_T} \\
&\quad + 2\Big(\partial_t u, \sum_{j>0} (\partial_j \tau_j) \partial_t u\Big)_{\Omega^T} + (\alpha_{0(0)} u, u)_{\Omega_0}^{\Omega_T}.
\end{aligned}$$

Hence

$$\begin{aligned}
2(\partial_t u, \mathfrak{M}u)_{\Omega^T} &= Q(t; u)\big|_0^T + \sum_{j=1}^{k} (\partial_t u, D_k u)_{\Omega_k^T} \\
&\quad + Q^0(t; u)\big|_0^T + 2\Big(\partial_t u, \Big(\sum_{j>0} \partial_j \tau_j\Big) \partial_t u\Big)_{\Omega^T}
\end{aligned} \tag{8.2}$$

where

$$Q(t; u) = \int_{\Omega_t} Q(t, x; \partial_t u, \nabla_x u)\, dx,$$

$$D_k = \sum_{j>0} \alpha_{jk} \alpha_j - 2\tau_k \tau - \alpha_{0k},$$

and $|Q^0(t, u)| \leqslant C \|u\|_{\Omega_t} \|\nabla_{x,t} u\|_{\Omega_t}$, with C fixed and $\|u\|_\Omega^2 = (u, u)_\Omega$.

We assign Cauchy data on Ω_0:

$$u(0, x) = u_0(x), \qquad \partial_t u(0, x) = u_1(x),$$

and on each Γ_k, a boundary condition of one of the forms

$$\begin{aligned} &\text{(a)} \quad u_t = 0, \\ &\text{(b)} \quad D_k u = \Psi_k(x)u_t + \nu_k(x)u - g_k(x, t), \\ &\text{(c)} \quad D_k u = C_k \partial_t u + \Psi_k(x)\partial_t u + \nu_k(x)u - g_k(x, t), \end{aligned} \tag{8.3}$$

where C_k is a tangential first order operator:

$$C_k = c_{k0}(x)\partial_t + \sum_{l \neq k} c_{kl}(x)\partial_l,$$

and $\Psi_k \geqslant 0$; here c_{k0}, c_{kl}, Ψ_k, ν_k are all assumed to be in $\mathcal{C}^1$, uniformly in x. Along $\Gamma_{jk}^+ = \bigcup_{T>0} \Gamma_{jk}^T$, $1 \leqslant j,k \leqslant N$, we assume that $c_{jk} + c_{kj}$ is either identically zero, or is bounded away from zero. Define

$$\Gamma'^+ = \bigcup \{(t, x) \in \Gamma_{jk}^+ : c_{jk} + c_{kj} < 0\},$$

$$\Gamma''^+ = \bigcup \{(t, x) \in \Gamma_{jk}^+ : c_{jk} + c_{kj} > 0\},$$

$$\Gamma'^{(T)} = \Gamma'^+ \cap \Gamma^T, \qquad \Gamma''^{(T)} = \Gamma''^+ \cap \Gamma^T,$$

and let $|c_{jk} + c_{kj}| > c_+ > 0$ on $\Gamma'^+ \cap \Gamma''^+$. Along Γ''^+ we assign

$$(c_{jk} + c_{kj})u_t = h(x). \tag{8.4}$$

Substitution of the boundary condition (8.3) into (8.2) gives

$$\begin{aligned} 2(\partial_t u, Mu)_{\Omega^T} = Q(t; u)\big|_0^T &+ \sum_{j=1}^{k} (\partial_t u, \beta_k u - g)_{\Gamma_k} \\ &+ Q^0(t_j u)\big|_0^T + 2\left(\partial_t u, \sum_{j>0}^{k} (\partial_j \tau_j)\partial_t u\right)_{\Omega^T}, \end{aligned} \tag{8.5}$$

where

$\beta_k = (C_k + \psi(x))\partial_t + \nu_k(x)$, in the cases (8.3)(b),(c),

$\beta_k = 0$ for the boundary condition (8.3)(a).

Integration by parts gives

$$\begin{aligned} (\partial_t u, \beta_k u)_{\Gamma_k} = (\partial_t u, \gamma_k \partial_t u)_{\Gamma_k^T} &+ \tfrac{1}{2}\{(\partial_t u, c_{k0}\partial_t u)_{\Gamma_{k,(T)}} - (u_t, c_{k,0}u_t)_{\Gamma_{k,(0)}} \\ &+ (u, \nu_k u)_{\Gamma_{k(T)}} - (u, \nu_k u)_{\Gamma_{k,(0)}}\}, \end{aligned} \tag{8.6}$$

where

$$\gamma_k = \Psi_k - \frac{1}{2} \sum_{k \neq l > 0} (\partial_l c_{kl}) \quad \text{and} \quad \Gamma_{k,(t)} = \Gamma_k^+ \cup \partial\Omega_t. \tag{8.7}$$

We add the assumptions

$$\gamma_k(x) \geqslant \gamma_0 \geqslant 0, \qquad \gamma_0 \text{ fixed}, \tag{8.8}$$

$$\text{if } \gamma_0 = 0, \qquad g \equiv 0. \tag{8.9}$$

Let us substitute (8.6) into (8.5) and apply Schwarz' inequality on the left-hand side of (8.5). The result can be written

$$\begin{aligned} Q(T; u) &+ (\partial_t u, \gamma\partial_t u)_{\Gamma^T} + Q^0(T; u) \\ &+ \tfrac{1}{2}(\partial_t u, c_{k0}\partial_t u)_{\Gamma_{k,(T)}} + \tfrac{1}{2}(\partial_t u, (c_{kl} + c_{lk})\partial_t u)_{\Gamma'(T)} + (u, \nu_k u)_{\Gamma_{k,(T)}} \\ &\leq C_1(\partial_t u, \partial_t u)_{\Omega_T} + C_2(\mathfrak{M}u, \mathfrak{M}u)_{\Omega_T} + \varepsilon(\partial_t u, \partial_t u)_{\Gamma^T} \\ &+ \varepsilon^{(-1)}(g, g)_{\Gamma^T} + Q(0; u) + Q^0(0; u) \\ &+ \tfrac{1}{2}(\partial_t u, c_{k,0}u_t)_{\Gamma_{k,(0)}} + (u, \nu_k u)_{\Gamma_{k(0)}} + \tfrac{1}{2}(\partial_t u, h)_{\Gamma''(T)}, \end{aligned} \tag{8.10}$$

where $\gamma_0 > \varepsilon > 0$ if $\gamma_0 > 0$, $\varepsilon = 0$ if $\gamma_0 = 0$, and $\varepsilon^{(-1)} = \varepsilon^{-1}$ if $\varepsilon > 0$, $\varepsilon^{(-1)} = 0$ if $\varepsilon = 0$. In the sequel, ε will be held fixed.

Denote the left-hand side of (8.10) by $\tilde{Q}_0^2(T; u)$, and set

$$Q_{\varepsilon,\lambda}(T_j u) = \tilde{Q}_0(T_j u) - \varepsilon(\partial_t u, \partial_t u) + \lambda(u, u)_{\Omega_T}, \qquad \lambda \in R.$$

Because of the estimate

$$|Q^0(T_j u)| + |(u, \nu_k u)_{\Gamma_{k,(T)}}| \leq \varepsilon Q(T; u) + c\hat{\varepsilon}^{-1}(u, u)_{\Omega_T}, \qquad 0 < \varepsilon < 1, \tag{8.11}$$

$Q_{\varepsilon,\lambda}$ is strictly positive for λ sufficiently large, say, $\lambda \geq \lambda_0$, in the sense that

$$\begin{aligned} Q_{\varepsilon,\lambda}(T_j u) \geq \tilde{q}_1(\partial_t u, \partial_t u)_{\Omega^T} &+ \tilde{q}_2(\nabla_x u, \nabla_x u)_{\Omega^T} \\ &+ \tilde{q}_3(\partial_t u, \partial_t u)_{\Gamma'(T)} + \tilde{q}_4(\partial_t u, \partial_t u)_{\Gamma_{k,(T)}}, \end{aligned} \tag{8.12}$$

with $\tilde{q}_1$, $\tilde{q}_2$ and $\tilde{q}_4 > 0$, and some $\tilde{q}_3 > \varepsilon(1 - \varepsilon)$.

Because of (8.12) and the facts that

$$\|u\|^2_{\Omega_T} - \|u\|^2_{\Omega_0} \leq \|u\|^2_{\Omega^T} + \|\partial_\varepsilon u\|^2_{\Omega^T},$$

and

$$(u, hu)_{\Gamma''} \geq c_+^{-1}(h, h)_{\Gamma''},$$

(8.10) implies also

$$Q_{\varepsilon,\lambda}(T_j u) \leq Q_{\varepsilon,\lambda}(0; u) + C^1_{\varepsilon,\lambda}\int_0^T Q_{\varepsilon,\lambda}(t; u)\,dt + C^2_{\varepsilon,\lambda}\int_0^T \mathfrak{F}(t)\,dt, \tag{8.13}$$

where

$$\mathfrak{F}(t) = (f, f)_{\Omega_t} + (g, g)_{\Gamma_t} + (h, h)_{\Gamma''_t},$$

with $f = Mu$, $\Gamma'_t = \Gamma'_t \cap \partial\Omega_t$, $\Gamma''_t = \Gamma'' \cap \partial\Omega_t$.

Hence we can estimate by Gronwall's inequality

$$Q_{\varepsilon,\lambda}(T; u) \leq (\exp C^1_{\varepsilon,\lambda}T)\left\{C^2_{\varepsilon,\lambda}\int_0^T \exp(-C^1_{\varepsilon,\lambda}\tau)\mathfrak{F}(\tau)\,d\tau + Q_{\varepsilon,\lambda}(0; u)\right\}, \qquad T \geq 0. \tag{8.14}$$

with any $\sigma > -C^1_{\varepsilon,\lambda}$, multiply both sides of (8.14) by $\exp((-\sigma - C^1_{\varepsilon,\lambda})T)\,dT$, and integrate from 0 to ∞; this leads to

$$\int_0^\infty \exp(-(\sigma + C^1_{\varepsilon,\lambda})T)Q_{\varepsilon,\lambda}(T; u)\,dT$$
$$\leqslant \sigma^{-1}C^2_{\varepsilon,\lambda}\left\{Q_{\varepsilon,\lambda}(0; u) + \int_0^\infty \exp((-\sigma - C^1_{\varepsilon,\lambda})T)\mathcal{F}(T)\,dT\right\}. \tag{8.15}$$

Since the equation and boundary condition have real coefficients, and since $Q_{\varepsilon,\lambda}$ is a real symmetric form, the inequalities (8.14) and (8.15) extend immediately to complex valued u provided we redefine $Q_{\varepsilon,\lambda}$ and $\mathcal{F}$ with conjugate bilinear forms replacing real quadratic forms. In the sequel we shall assume this has been done.

In summary, we have

THEOREM 8.1. *Let $\mathfrak{M}$ be a second order hyperbolic operator as described above. Let $u \in \mathcal{C}^2_0(R^{M+N+1})$ satisfy boundary condition along Γ^+ and Γ'' given by* (8.3) *and* (8.4), *and assume that* (8.1), (8.8) *and* (8.9) *hold. Then u satisfies* (8.14) *with any $T > 0$, and* (8.15).

REMARK. In the above analysis, we could have permitted ν_j to depend on t, $j = 1, \dots, N$; the only difference would be the appearance of an additional term in the right-hand side of (8.10), namely

$$\frac{1}{2}\sum_k (u, (\partial_t \nu_k)u)_{\Gamma^T_k}.$$

Since this term can be estimated by $C\int_0^T Q(t; u)\,dt$, estimates of the form of (8.13) and (8.14) would again hold.

9. The hyperbolic case. Existence of strong solutions.

THEOREM 9.1. *Let $\mathfrak{M}$ be as in* §8, *and assume either that $N = 2$, or that for the time reduced problem, the hypotheses of Theorem* 3.1 *hold. Then the equation* (8.10) *with boundary conditions*

$$\begin{aligned} &\text{(a)} \quad \mathfrak{M}u = f \quad \text{in } \Omega^+,\\ &\text{(b)} \quad u = 0 \text{ or } Bu = g \quad \text{on } \Gamma^+,\\ &\text{(c)} \quad B''u = h \quad \text{on } \Gamma''^+, \end{aligned} \tag{9.1}$$

with

$$B = -D_k + C_k\partial_t + \Psi_k\partial_t + \nu_k, \qquad g = g_k \quad \text{on } \Gamma^+_k,$$
$$B'' = c_{jk} + c_{kj} \quad \text{on } \Gamma''^+,$$

has a strong solution in satisfying (8.14) *and* (8.15) *with $\mathcal{F}$ as given below equation* (8.13), *provided* (8.1), (8.8) *and* (8.9) *hold, and provided the right-hand sides of* (8.14) *and* (8.15) *are finite. Here λ, ε, $C^1_{\varepsilon,\lambda}$ and $C^2_{\varepsilon,\lambda}$ depend on B and $\mathfrak{M}$, but are independent of u.*

By a strong solution, we mean one which can be approximated in the graph norm by functions $u_\delta \in C^\infty(R^{N+M+1})$ such that for all $T > 0$, the restriction of the support of u_δ to $[0, T] \times R^{N+1}$ is bounded.

PROOF. Abbreviate (9.1) as $Tu = F$. Since smooth data are dense, it suffices to consider data in $\mathcal{C}_0^\infty$ and homogeneous initial conditions;

$$u(0, t) = \partial_t u(0, t) = 0.$$

We solve by taking a Fourier transformation in t:

$$(2\pi)^{-1}\int_{-\infty}^{\infty} e^{-it\tau}\hat{u}(t, \cdot)\, dt = \hat{u}(\tau, \cdot), \qquad \operatorname{Im} \tau \leqslant 0,$$

where u has been extended to vanish for $t > 0$. This puts the problem (9.1) into the form

$$\begin{aligned} \mathfrak{M}_\tau \hat{u} &= \hat{F} \quad \text{in } \Omega, \\ B_\tau \hat{u} &= \hat{g} \quad \text{on } \Gamma, \\ B''_\tau \hat{u} &= \hat{h} \quad \text{on } \Gamma'', \end{aligned} \tag{9.2}$$

where $\mathfrak{M}_\tau = e^{-it\tau}Me^{it\tau}$; note that (9.2) is an elliptic problem. If $\operatorname{Re} \tau = 0$, then (9.2) is a problem of the type considered in §§1-5, except possibly for scalar multiples in the leading terms of B_τ and of B''_τ. We abbreviate (9.2) as

$$T_\varepsilon \hat{u} = F_\tau, \quad F_\tau = \{\hat{f}(\tau; \cdot), \hat{g}(\tau; \cdot), h(\tau; \cdot)\}. \tag{9.3}$$

With

$$\operatorname{Im} \tau = \sigma < \min(0, C'_{\varepsilon,\lambda}/2), \tag{9.4}$$

define

$$u_\tau(t, \cdot) = e^{it\tau}\hat{u}(\tau, \cdot),$$

and note that

$$e^{2\sigma t}Q_{\varepsilon,\lambda}(t; u_\sigma) \overset{\text{def}}{=} Q_\sigma \hat{u}(\tau, \cdot). \tag{a}$$

is independent of t, and

$$Tu_\tau(t; \cdot) = e^{it\tau}T_\tau \hat{u}(\tau,). \tag{b}$$

Let us apply the inequality (8.14) to the function u_τ with t in place of T, and let $t \to +\infty$. In the limit, using (a) and (b), we find

$$Q_\sigma \hat{u}(\tau, \cdot) \leqslant C^2_{\varepsilon,\lambda}(C'_{\varepsilon,\lambda} + 2\sigma)^{-1}\|F_\tau\|^2. \tag{9.5}$$

On the other hand, because of Theorems 1.1, 2.1, and 5.1, if σ is sufficiently large negative, say $0 < -|\sigma_0|$, the equation

$$T_{i\sigma} v = F_{i\sigma}$$

has a solution $v_{i\sigma}$ satisfying 1.15 and, if applicable, (1.16). The identity of $\mathfrak{L}_\omega$ with $\mathfrak{L}_s$ (Theorem 6.1) implies that $v_{i\sigma}$ admits $\mathfrak{L}_s$ and satisfies (9.4). By the continuation method, then, we can establish the existence of $u_{(\tau)}$ for any τ satisfying (9.4); moreover, $u_{(\tau)}$ satisfies (9.5).

We claim that the inverse Fourier transform u of $u_{(\tau)}$ satisfies our problem:

$$u(t, \cdot) = \int e^{it(\mu + i\sigma)}u_{(\mu + i\sigma)}(\cdot)\, d\mu,$$

where σ satisfies (9.4). That $u_{(\mu+i\sigma)}$ satisfies the equation and boundary condition is immediate. We need to verify that u has support in $t \geqslant 0$ and that u has zero

Cauchy data. Clearly $\hat{u}(\tau) = u_{(\tau)}$ is holomorphic in τ for $\operatorname{Im} \tau < -|\sigma_0|$. Because

$$\int |F_{\mu+i\sigma}|^2 \, d\mu \leqslant \int |F(t)|^2 \, dt$$

uniformly for σ obeying (9.4), we conclude from (9.5) together with the one-sided Paley-Wiener theorem that $Q_{\varepsilon,\lambda}(t; u) \equiv 0$ for $t < 0$. Finally, the smoothness of the data implies that Q varies smoothly in t, so that $Q_{\varepsilon,\lambda}(0; u) = 0$, and hence u has zero Cauchy data.

REMARK. We might equally well have proceeded by constructing a weak solution in $\{(t, x) \in \Omega^+ : 0 \leqslant t \leqslant T\}$ by variational methods as previously.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195